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Polynomials Associated with Equilibrium Positions in Calogero-Moser Systems

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Abstract

In a previous paper (Corrigan-Sasaki), many remarkable properties of classical Calogero and Sutherland systems at equilibrium are reported. For example, the minimum energies, frequencies of small oscillations and the eigenvalues of Lax pair matrices at equilibrium are all "integer valued". The equilibrium positions of Calogero and Sutherland systems for the classical root systems (A_r, B_r, C_r and D_r) correspond to the zeros of Hermite, Laguerre, Jacobi and Chebyshev polynomials. Here we define and derive the corresponding polynomials for the exceptional (E_6, E_7, E_8, F_4 and G_2) and non-crystallographic ($I_2(m)$, H_3 and H_4) root systems. They do not have orthogonality but share many other properties with the above mentioned classical polynomials.

1 Introduction

The relationship between classical and quantum integrability has fascinated many physicists and mathematician. In a recent paper by Corrigan and Sasaki [1], this issue has been extensively investigated in the framework of Calogero-Moser systems [2, 3, 4]. One major result is that certain "quantised" information seems to be encoded in the classical system. For example, the eigenvalues of classical Lax pair matrices at the equilibrium points are "integer valued" [1]. The connection between the zeros of Hermite and Laguerre polynomials and the equilibrium points of A_r and B_r (D_r) Calogero systems has been known for many years [5, 6, 7]. In [1] it is found out that the zeros of Jacobi polynomials are related to the equilibrium points of BC_r (D_r) Sutherland system. In the present paper we define and derive the polynomials associated with the equilibrium points of the other Calogero and Sutherland systems. These are associated with Calogero systems based on non-crystallographic root systems, Calogero and Sutherland systems based on the exceptional root systems and the A_r Sutherland systems. The Chebyshev polynomials (5.3) are associated with the A_r Sutherland systems.

In general, the polynomials are determined by the potential, $q^2 + 1/q^2$ (the Calogero system [2]) and $1/\sin^2 q$ (the Sutherland system [3]), the root system Δ and the set of weights $\mathcal{R}$. For the classical root systems and for the (non-trivial) smallest dimensional $\mathcal{R}$, that is the set of vector weights $\mathbf{V}$ or the set of short roots Δ_S, the polynomials turn out to be classical *orthogonal* polynomials; Hermite, Laguerre, Jacobi and Chebyshev polynomials [8]. The orthogonality does not hold for the polynomials for the exceptional root systems and for the classical root systems with generic $\mathcal{R}$'s. Like their classical counterparts, these new polynomials have "integer coefficients" only, if multiplied by a certain factor. In most cases, it is possible to define the polynomials to be *monic* (that is, the highest degree term has *unit* coefficient) and *integer coefficients only*. Some polynomials are too lengthy to be displayed in the paper; an E_8 polynomial has 121 terms and its typical integer coefficient has about 150 digits. They are presented in [9]. Some root systems are related by Dynkin diagram foldings; $A_{2r-1} \to C_r$, $D_{r+1} \to B_r$, $E_6 \to F_4$ and $D_4 \to G_2$. These imply relations among the corresponding Calogero-Moser systems at certain ratio of the coupling constants. These, in turn, imply relations among the corresponding polynomials, which are determined independently. These relations are either identities among classical polynomials, many of which are “new” in the sense they are not listed in standard mathematical textbooks [8], or they provide non-trivial checks for the newly derived polynomials. The significance and other detailed properties of these new polynomials deserve further study.

This paper is organised as follows. In section two a brief introduction of Calogero-Moser systems is given to set the stage and notation. Equations for determining equilibrium points are discussed in some detail. In section three Coxeter (Weyl) invariant polynomials associated with equilibrium positions are introduced for a set of weights $\mathcal{R}$ for Calogero and Sutherland systems. For the rational potential (Calogero systems) the definition is almost unique, whereas we have several choices of the definitions of the polynomials for the trigonometric potential (Sutherland system). Section four and five are the main body of the paper. The Coxeter (Weyl) invariant polynomials are determined and presented for all root systems Δ and for major choices of $\mathcal{R}$'s for Calogero (section four) and Sutherland systems (section five). Section six is for summary and comments. We will present a heuristic argument for deriving the classical orthogonal polynomials starting from the pre-potentials (2.4) of Calogero and Sutherland systems.

2 Equilibrium in Calogero-Moser System

Let us start with a brief introduction of Calogero-Moser systems [2, 3, 4]. We stick to the notation of a recent paper [1], unless otherwise mentioned. Calogero-Moser systems are integrable multiparticle dynamical systems at the classical as well as quantum levels. They have a long range potential (rational, trigonometric, hyperbolic and elliptic) and the integrable multiparticle interactions are governed by the root systems [10]. Classical integrability through Lax formalism is known for all potentials for the classical root systems [10] as well as for the exceptional [11, 12] and non-crystallographic [12] root systems. Quantum integrability of the systems having degenerate potentials (rational, trigonometric and hyperbolic) is now systematically understood for all root systems in terms of Dunkl operator formalism [13, 14] and the quantum Lax pair formalism [15, 16]. To a system of r particles in one dimension, we associate a root system Δ of rank r. This is a set of vectors in $\mathbb{R}^r$ invariant under reflections in the hyperplane perpendicular to each vector in Δ:

$$\Delta \ni s_\alpha(\beta) = \beta - (\alpha^\vee \cdot \beta)\alpha, \quad \alpha^\vee = \frac{2\alpha}{\alpha^2}, \quad \alpha, \beta \in \Delta. \tag{2.1}$$

The set of reflections $\{s_\alpha |\, \alpha \in \Delta\}$ generates a finite reflection group G_Δ, known as a Coxeter (or Weyl) group. Among Calogero-Moser systems the Calogero systems (with $q^2 + 1/q^2$ potential) and the Sutherland systems (with $1/\sin^2 q$ potential) have discrete energy eigenvalues only when quantised. The Calogero and Sutherland systems have equilibrium positions, which are characterised by two equivalent ways [1]. That is where the classical potential takes the absolute minimum and simultaneously the groundstate wavefunction takes the absolute maximum. At the equilibrium positions of the Calogero and Sutherland systems, associated spin exchange models are defined for each root system [17]. The best known example is the Haldane-Shastry model which is based on A_r Sutherland systems [18]. The integrability and the well ordered spectrum of the spin exchange models are closely related with the special properties of systems at equilibrium [1].

The classical Hamiltonians of the Calogero and Sutherland systems read[1]:

$$\mathcal{H}_C = \frac{1}{2}p^2 + V_C, \quad V_C = \begin{cases} \dfrac{\omega^2}{2}q^2 + \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} \dfrac{g_\rho^2 \rho^2}{(\rho\cdot q)^2}\,, \\ \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} \dfrac{g_\rho^2\rho^2}{\sin^2(\rho\cdot q)}\,. \end{cases} \tag{2.2}$$

In these formulae, Δ_+ is the set of positive roots and $\omega > 0$ is the angular frequency of the confining harmonic potential, $g_\rho > 0$ are real coupling constants which are defined on orbits of the corresponding Coxeter group, *i.e.,* they are identical for roots in the same orbit. The classical potential V_C can be written succinctly in terms of a *pre-potential* W [15]:

$$V_C = \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W}{\partial q_j}\right)^2 + \tilde{\mathcal{E}}_0, \tag{2.3}$$

in which

$$W = \begin{cases} -\dfrac{\omega}{2}q^2 + \displaystyle\sum_{\rho\in\Delta_+} g_\rho \log|\rho\cdot q|\,, \\ \displaystyle\sum_{\rho\in\Delta_+} g_\rho \log|\sin(\rho\cdot q)|\,, \end{cases} \tag{2.4}$$

and $\tilde{\mathcal{E}}_0$ is the minimum energy. Let us recall that the pre-potential W is related to the ground state wavefunction of the quantum theory ϕ_0 by $\phi_0 = e^W$ (eq.(2.6) of [1]) and that W, V_C

[1]For $\Delta = BC_r$ the trigonometric potential should read $g_M^2 \sum_{\rho\in\Delta_{M+}} 1/\sin^2(\rho\cdot q) + 2g_L^2 \sum_{\rho\in\Delta_{L+}} 1/\sin^2(\rho\cdot q) + g_S(g_S+2g_L)/2 \sum_{\rho\in\Delta_{S+}} 1/\sin^2(\rho\cdot q)$, with $\rho_M^2 = 2$, $\rho_L^2 = 4$ and $\rho_S^2 = 1$.

and $\mathcal{H}_C$ are Coxeter (Weyl) invariant:

$$\mathcal{H}_C(p,q) = \mathcal{H}_C(s_\alpha(p), s_\alpha(q)), \quad W(q) = W(s_\alpha(q)), \quad V_C(q) = V_C(s_\alpha(q)) \quad (\forall \alpha \in \Delta). \tag{2.5}$$

The classical equilibrium point

$$p = 0, \quad q = \bar{q} \tag{2.6}$$

is determined by the equations [1]

$$\left.\frac{\partial V_C}{\partial q_j}\right|_{\bar{q}} = 0 \ \text{ or equivalently } \ \left.\frac{\partial W}{\partial q_j}\right|_{\bar{q}} = 0 \quad (j = 1, \ldots, r). \tag{2.7}$$

In other words, it is a *minimal* point of the classical potential V_C, and simultaneously it is a *maximal* point of the pre-potential W and of the ground state wavefunction $\phi_0 = e^W$, since the matrix determining the frequencies of small oscillations around the equilibrium

$$\left.\frac{\partial^2 W}{\partial q_j \partial q_l}\right|_{\bar{q}} \quad (j, l = 1, \ldots, r), \tag{2.8}$$

is negative definite [1]. The equilibrium points are not unique. There is one equilibrium point in each Weyl chamber (alcove) [1], that is if $\bar{q}$ is an equilibrium point, so is $s_\rho(\bar{q})$, $\forall \rho \in \Delta$, due to the Coxeter (Weyl) invariance of W (2.5). It is also easy to see that if $\bar{q}$ is an equilibrium point, so is $-\bar{q}$.

The equilibrium equation for the pre-potential W, for Calogero systems based on *simply laced* root systems, that is A_r, D_r, E_r, I_2(odd) and H_r, reads:

$$\sum_{\rho \in \Delta_+} \frac{\rho}{\rho \cdot \bar{q}} = \frac{\omega}{g} \bar{q}.$$

If we define a *rescaled equilibrium point* by

$$\tilde{q} \equiv \sqrt{\frac{\omega}{g}}\, \bar{q}, \tag{2.9}$$

it satisfies a simple equation independent of the coupling constant:

$$\sum_{\rho \in \Delta_+} \frac{\rho}{\rho \cdot \tilde{q}} = \tilde{q}. \tag{2.10}$$

For Calogero systems based on *non-simply laced* root systems, that is B_r, C_r, F_4, G_2 and I_2(even)[2], the equation reads:

$$\sum_{\rho \in \Delta_{L+}} \frac{\rho}{\rho \cdot \bar{q}} + k \sum_{\rho \in \Delta_{S+}} \frac{\rho}{\rho \cdot \bar{q}} = \frac{\omega}{g_L} \bar{q}, \qquad k \equiv \frac{g_S}{g_L}.$$

[2]For I_2(even) we have $k \equiv g_e/g_o$.

Again a *rescaled equilibrium point*

$$\tilde{q} \equiv \sqrt{\frac{\omega}{g_L}}\,\bar{q}, \tag{2.11}$$

satisfies a simple equation depending only on the ratio of the two coupling constants g_S and g_L:

$$\sum_{\rho\in\Delta_{L+}} \frac{\rho}{\rho\cdot\tilde{q}} + k \sum_{\rho\in\Delta_{S+}} \frac{\rho}{\rho\cdot\tilde{q}} = \tilde{q}, \qquad k \equiv \frac{g_S}{g_L}. \tag{2.12}$$

As is clear from (2.10) and (2.12), the equilibrium point $\tilde{q}$ ($\bar{q}$) is independent of the normalisation of roots in Δ.

The situation is simpler in the Sutherland systems which do not have an extra parameter ω. For crystallographic *simply laced* root systems, that is A_r, D_r and E_r, the equation for $\bar{q}$ is independent of the coupling constant:

$$\sum_{\rho\in\Delta_+} \rho\cot(\rho\cdot\bar{q}) = 0. \tag{2.13}$$

For crystallographic *non-simply laced* root systems, that is B_r, C_r, F_4 and G_2, the equation for $\bar{q}$ depends only on the ratio of the two coupling constants g_S and g_L:

$$\sum_{\rho\in\Delta_{L+}} \rho\cot(\rho\cdot\bar{q}) + k \sum_{\rho\in\Delta_{S+}} \rho\cot(\rho\cdot\bar{q}) = 0, \qquad k \equiv \frac{g_S}{g_L}. \tag{2.14}$$

For the BC_r system, which has *three* coupling constants g_S, g_M and g_L for the short, middle and long roots, the equation depends on two coupling ratios:

$$\sum_{\rho\in\Delta_{M+}} \rho\cot(\rho\cdot\bar{q}) + k_1 \sum_{\rho\in\Delta_{S+}} \rho\cot(\rho\cdot\bar{q}) + k_2 \sum_{\rho\in\Delta_{L+}} \rho\cot(\rho\cdot\bar{q}) = 0, \quad k_1 \equiv \frac{g_S}{g_M}, \quad k_2 \equiv \frac{g_L}{g_M}. \tag{2.15}$$

3 Polynomials

Here we give the general definitions of the Coxeter (Weyl) invariant polynomials associated with equilibrium positions in Calogero and Sutherland systems. Naturally, the definitions for the Calogero systems are different from those for the Sutherland systems except for the common features that the polynomials are Coxeter (Weyl) invariant and are specified by the root system Δ and a set of D vectors $\mathcal{R}$

$$\mathcal{R} = \{\mu^{(1)},\ldots,\mu^{(D)}\,|\,\mu^{(a)}\in\mathbb{R}^r\}, \tag{3.1}$$

which form a single orbit of the corresponding reflection (Weyl) group G_Δ. The set of values at the equilibrium, $\{\mu\cdot\bar{q}\,|\,\mu\in\mathcal{R}\}$, is Coxeter (Weyl) invariant. In this paper we consider only such $\mathcal{R}$'s that are customarily used for Lax pairs. They are the set of roots Δ itself for simply laced root systems, the set of long (short, middle) roots Δ_L (Δ_S, Δ_M) for non-simply laced root systems and the so-called sets of *minimal weights*. The latter is better specified by the corresponding fundamental representations, which are all the fundamental representations of A_r, the vector ($\mathbf{V}$), spinor ($\mathbf{S}$) and conjugate spinor ($\bar{\mathbf{S}}$) representations of D_r and $\mathbf{27}$ ($\overline{\mathbf{27}}$) of E_6 and $\mathbf{56}$ of E_7.

For Calogero systems the definition is rather unique and it is given by

$$P_\Delta^{\mathcal{R}}(k|x)=\prod_{\mu\in\mathcal{R}}(x-\mu\cdot\tilde{q}), \tag{3.2}$$

in which k denotes the possible dependence on the ratio of the coupling constants, for the systems based non-simply laced root systems (2.12). It should be noted that the above polynomial depends on the normalisation of the vectors $\mu\in\mathcal{R}$ implicitly. Changing $\mathcal{R}\to c\mathcal{R}$ ($\mu\to c\mu$) can be absorbed by rescaling of x:

$$P_\Delta^{c\mathcal{R}}(k|x)=\prod_{\mu\in\mathcal{R}}(x-c\mu\cdot\tilde{q})=c^D P_\Delta^{\mathcal{R}}(k|x/c). \tag{3.3}$$

For Sutherland systems we have several candidates for polynomials:

$$P_{\Delta,\,s}^{\mathcal{R}}(k|x)=\prod_{\mu\in\mathcal{R}}\Big(x-\sin(\mu\cdot\bar{q})\Big),\quad P_{\Delta,\,s2}^{\mathcal{R}}(k|x)=\prod_{\mu\in\mathcal{R}}\Big(x-\sin(2\mu\cdot\bar{q})\Big), \tag{3.4}$$

$$P_{\Delta,\,c}^{\mathcal{R}}(k|x)=\prod_{\mu\in\mathcal{R}}\Big(x-\cos(\mu\cdot\bar{q})\Big),\quad P_{\Delta,\,c2}^{\mathcal{R}}(k|x)=\prod_{\mu\in\mathcal{R}}\Big(x-\cos(2\mu\cdot\bar{q})\Big), \tag{3.5}$$

in which k denotes possible dependence on the ratio(s) of coupling constants, as before. Not all of them give interesting objects, as we will see presently. In all cases the polynomials are monic and of degree D.

In case $\mathcal{R}$ is *even*, that is,

$$\mu\in\mathcal{R}\Longleftrightarrow-\mu\in\mathcal{R}, \tag{3.6}$$

then sometimes it is advantageous to consider $P_\Delta^{\mathcal{R}}(k|x)$, $P_{\Delta,\,s}^{\mathcal{R}}(k|x)$ and $P_{\Delta,\,s2}^{\mathcal{R}}(k|x)$ as polynomials in $y\equiv x^2$ of degree $D/2$:

$$\prod_{\mu\in\mathcal{R}_+}\left(y-(\mu\cdot\tilde{q})^2\right),\quad \prod_{\mu\in\mathcal{R}_+}\Big(y-\sin^2(\mu\cdot\bar{q})\Big),\quad \prod_{\mu\in\mathcal{R}_+}\Big(y-\sin^2(2\mu\cdot\bar{q})\Big), \tag{3.7}$$

in which $\mathcal{R}_+$ is the *positive* part of $\mathcal{R}$. In this case the "cosine" polynomials $P^{\mathcal{R}}_{\Delta,\,c(2)}(k|x)$, (3.5) should better be redefined as

$$P^{\mathcal{R}_+}_{\Delta,\,c}(k|x)=\prod_{\mu\in\mathcal{R}_+}\Big(x-\cos(\mu\cdot\bar{q})\Big),\quad P^{\mathcal{R}_+}_{\Delta,\,c2}(k|x)=\prod_{\mu\in\mathcal{R}_+}\Big(x-\cos(2\mu\cdot\bar{q})\Big),\tag{3.8}$$

since the original polynomials (3.5) are the squares of the new ones. It is easy to see that $P^{\mathcal{R}}_{\Delta,\,s}(k|y)$ and $P^{\mathcal{R}_+}_{\Delta,\,c2}(k|x)$ are equivalent:

$$P^{\mathcal{R}}_{\Delta,\,s}(k|x)=(-2)^{-D/2}P^{\mathcal{R}_+}_{\Delta,\,c2}(k|1-2x^2).\tag{3.9}$$

Likewise, for *even* $\mathcal{R}$, $P^{\mathcal{R}}_{\Delta,\,s2}(k|x)$ is a "square" of $P^{\mathcal{R}_+}_{\Delta,\,c2}(k|x)$:

$$\begin{aligned}P^{\mathcal{R}}_{\Delta,\,s2}(k|x) &= \prod_{\mu\in\mathcal{R}}\Big(x-\sin(2\mu\cdot\bar{q})\Big)=\prod_{\mu\in\mathcal{R}_+}\Big(x^2-\sin^2(2\mu\cdot\bar{q})\Big)\\ &= \prod_{\mu\in\mathcal{R}_+}\Big(u-\cos(2\mu\cdot\bar{q})\Big)\Big(-u-\cos(2\mu\cdot\bar{q})\Big),\quad u^2\equiv 1-x^2\\ &= P^{\mathcal{R}_+}_{\Delta,\,c2}(k|u)P^{\mathcal{R}_+}_{\Delta,\,c2}(k|-u).\end{aligned}\tag{3.10}$$

The right hand side is an even polynomial in u, thus it is a polynomial in u^2 and in x^2. The change of variables $u\leftrightarrow x$ corresponds to the change in the character of the variables, $\cos\leftrightarrow\sin$. This imposes a quite non-trivial check for the $s2$ and $c2$ polynomials which are determined separately.

As shown in the following sections, the polynomials associated with the classical root systems (A_r, B_r, C_r and D_r) and $I_2(m)$ are either classical polynomials for the smallest dimensional $\mathcal{R}$ or those closely related to them, see for example, (4.32), (4.33), (5.41), (5.42). For the exceptional and non-crystallographic root systems, the equilibrium positions are evaluated numerically and the polynomials are obtained by rationalisation of the coefficients in terms of Mathematica. At each step, the result is verified by many consistency checks; the "integer eigenvalues" of the matrix (2.8) for the values of $\bar{q}$, the identities implied by Dynkin diagram foldings and identities (3.10) for the polynomials. Let us conclude this section with an important remark that these polynomials are independent of the specific representation of the root and weight vectors. In other words, the polynomials are Coxeter (Weyl) invariant.

4 Calogero Systems

Let us first discuss the systems based on the classical root systems.

4.1 A_r

The equations (2.10) for $\Delta = A_r$ read

$$\sum_{\substack{l=1\\ l\neq j}}^{r+1} \frac{1}{\tilde{q}_j - \tilde{q}_l} = \tilde{q}_j, \quad (j = 1, \ldots, r+1). \tag{4.1}$$

These determine $\{\tilde{q}_j = \sqrt{\frac{\omega}{g}}\bar{q}_j \,|\, j = 1, \ldots, r+1\}$ to be the zeros of the Hermite polynomial $H_{r+1}(x)$ [8], with the Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = 2^n x^n + \cdots. \tag{4.2}$$

If ordered by the value, $\tilde{q}_1 > \tilde{q}_2 > \cdots > \tilde{q}_{r+1}$ or reverse, they possess the symmetry

$$\tilde{q}_j = -\tilde{q}_{r+2-j}, \tag{4.3}$$

and especially $\tilde{q}_{(r+2)/2} = 0$ for r even. Thus we have

$$\tilde{q}_1 + \tilde{q}_2 + \cdots + \tilde{q}_{r+1} = 0. \tag{4.4}$$

4.1.1 $\mathcal{R} = \mathbf{V}$ for A_r

This case was reported by Calogero a quarter century ago [5]. The set of weights of the vector representation is

$$\mathbf{V} = \Big\{\mu_j \equiv \mathbf{e}_j - \frac{1}{r+1}\sum_{l=1}^{r+1} \mathbf{e}_l \;\Big|\; j = 1, \ldots, r+1\Big\}. \tag{4.5}$$

Throughout this paper we denote an orthonormal basis of $\mathbb{R}^r$ ($\mathbb{R}^{r+1}$ for A_r case) by $\{\mathbf{e}_j\}$.

In this case, we have $\mu_j \cdot \tilde{q} = \tilde{q}_j$ due to (4.4) and $\mu^2 = r/(r+1)$. The polynomial (3.2) is given by the Hermite polynomial

$$P_r^{\mathbf{V}}(x) \equiv P_{A_r}^{\mathbf{V}}(x) = \prod_{j=1}^{r+1} (x - \tilde{q}_j) = 2^{-(r+1)} H_{r+1}(x). \tag{4.6}$$

They are orthogonal to each other:

$$\int_{-\infty}^{\infty} P_r^{\mathbf{V}}(x) P_s^{\mathbf{V}}(x)\, e^{-x^2}\, dx \;\propto \delta_{r\,s}. \tag{4.7}$$

Needless to say, Hermite polynomials are of integer coefficients. It is interesting to note that another definition

$$P_{A_r}^{2\mathbf{V}}(x) = \prod_{j=1}^{r+1} (x - 2\tilde{q}_j) = H_{r+1}(x/2) = 2^{r+1} P_{A_r}^{\mathbf{V}}(x/2). \tag{4.8}$$

gives a *monic* polynomial with *all integer coefficients*.

4.1.2 $\mathcal{R} = \mathbf{V}_i$ for A_r

The set of weights of the i-th fundamental representation (i-th rank anti-symmetric tensor representation, $1 \le i \le r$) is

$$\mathbf{V}_i = \Big\{\mu_{j_1} + \cdots + \mu_{j_i} \,\Big|\, 1 \le j_1 < \cdots < j_i \le r+1\Big\}, \quad D = D_i \equiv \binom{r+1}{i}. \tag{4.9}$$

The above $\mathbf{V}$ (4.5) is $\mathbf{V} = \mathbf{V}_1$. In this case we have $\mu^2 = i(r+1-i)/(r+1)$. We can show that the polynomial (3.2)

$$P^{\mathbf{V}_i}_{A_r}(x) = \prod_{1\le j_1<\cdots<j_i\le r+1} \Big(x - (\tilde{q}_{j_1} + \cdots \tilde{q}_{j_i})\Big) = P^{\mathbf{V}_{r+1-i}}_{A_r}(x) \tag{4.10}$$

can be expressed in terms of the coefficients of $H_{r+1}(x)$ by the same method as given in section 4.2.5, and $P^{2\mathbf{V}_i}_{A_r}(x)$ gives a monic polynomial with integer coefficients.

Here we report only on $\mathbf{V}_2$ because it seems that the other representations ($3 \le i \le r-2$) do not provide any interesting results. (For lower rank r, the explicit forms of the polynomials $P^{\mathbf{V}_i}_{A_r}(x)$ can be found in [9].) Due to (4.3), eq.(4.10) becomes

$$\begin{aligned} P^{\mathbf{V}_2}_{A_r}(x) &= \prod_{1\le j<l\le r+1} \Big(x - (\tilde{q}_j + \tilde{q}_l)\Big) \qquad\qquad (4.11) \\ &= \begin{cases} x^{(r+1)/2} \displaystyle\prod_{1\le j<l\le (r+1)/2} \Big(x^2 - (\tilde{q}_j - \tilde{q}_l)^2\Big)\Big(x^2 - (\tilde{q}_j + \tilde{q}_l)^2\Big) & r\,:\,\text{odd} \\ x^{r/2} \displaystyle\prod_{j=1}^{r/2}(x^2 - \tilde{q}_j^2) \cdot \prod_{1\le j<l\le r/2} \Big(x^2 - (\tilde{q}_j - \tilde{q}_l)^2\Big)\Big(x^2 - (\tilde{q}_j + \tilde{q}_l)^2\Big) & r\,:\,\text{even.} \end{cases} \end{aligned}$$

Based on the fact that the zeros of Hermite and Laguerre polynomials are related as seen from the formulae (4.23), this can be expressed by using the polynomials associated with the B_r Calogero systems in the following way:

$$P^{\mathbf{V}_2}_{A_{2r-1}}(x) = x^r P^{\Delta_L}_{B_r}(1/2|x), \quad P^{\mathbf{V}_2}_{A_{2r}}(x) = x^{r-1} P^{\mathbf{V}}_{A_{2r}}(x) P^{\Delta_L}_{B_r}(3/2|x). \tag{4.12}$$

The explicit forms of the functions $P^{\Delta_L}_{B_r}(k|x)$ for lower r are given in section 4.2.5.

4.1.3 $\mathcal{R} = \Delta$ for A_r

We have $\Delta = \{\pm(\mathbf{e}_j - \mathbf{e}_l) \,|\, 1 \le j < l \le r+1\}$, $D = r(r+1)$ and $\mu^2 = 2$. The polynomial has a factorized form:

$$P^{\Delta}_{A_r}(x) = \prod_{1\le j<l\le r+1} \Big(x^2 - (\tilde{q}_j - \tilde{q}_l)^2\Big) = \begin{cases} x^2 - 2 & (r=1) \\ x^{-r-1} P^{2\mathbf{V}}_{A_r}(x)\Big(P^{\mathbf{V}_2}_{A_r}(x)\Big)^2 & (r \ge 2)\,. \end{cases} \tag{4.13}$$

Another definition $P^{2\Delta}_{A_r}(x)$ gives a monic polynomial with integer coefficients.

4.2 B_r and D_r

Assuming $\bar{q}_j \neq 0$, the equations (2.12) for $\Delta = B_r$ with $k \equiv g_S/g_L$ read

$$\sum_{\substack{l=1\\ l\neq j}}^{r} \frac{1}{\tilde{q}_j^2 - \tilde{q}_l^2} + \frac{k}{2\tilde{q}_j^2} = \frac{1}{2} \quad (j = 1, \ldots, r). \tag{4.14}$$

They determine $\{\tilde{q}_j^2 = \frac{\omega}{g_L}\bar{q}_j^2 \,|\, j = 1, \ldots, r\}$, as the zeros of the associated Laguerre polynomial $L_r^{(\alpha)}(x)$, with $\alpha = k - 1 = g_S/g_L - 1$ [1, 8, 10]. The Rodrigues' formula reads

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \left(\frac{d}{dx}\right)^n \left(e^{-x}\, x^{n+\alpha}\right) = \frac{(-1)^n}{n!} x^n + \cdots. \tag{4.15}$$

For the subcase with $g_S = 0$, that is $\Delta = D_r$, $\{\tilde{q}_j^2 = \frac{\omega}{g_L}\bar{q}_j^2 \,|\, j = 1, \ldots, r\}$, are the zeros of the associated Laguerre polynomial [8, 10],

$$r\, L_r^{(-1)}(x) = -x L_{r-1}^{(1)}(x), \tag{4.16}$$

for which one of the $\tilde{q}_j$ is zero. This also means that the $\{\tilde{q}_j^2\}$ of B_r for $g_S/g_L = 2$ or $\alpha = 1$ are the same as the non-vanishing $\{\tilde{q}_j^2\}$ of D_{r+1}. This can be understood easily from the Dynkin diagram folding $D_{r+1} \to B_r$. We omit C_r case, because C_r is obtained from B_r by interchanging the short (S) and long (L) roots.

4.2.1 $\mathcal{R} = \Delta_S$ for B_r

Since $\Delta_S = \{\pm \mathbf{e}_j \,|\, j = 1, \ldots, r\}$ is *even*, it is advantageous to consider the polynomials in $y \equiv x^2$, (3.7),

$$P_r^{\Delta_S}(y) \equiv P_{B_r}^{\Delta_S}(k|x) = \prod_{j=1}^{r} \left(x^2 - \tilde{q}_j^2\right) = (-1)^r r! L_r^{(\alpha)}(y), \quad \alpha = k - 1 = g_S/g_L - 1. \tag{4.17}$$

They are orthogonal to each other:

$$\int_0^\infty P_r^{\Delta_S}(y) P_s^{\Delta_S}(y)\, y^\alpha e^{-y}\, dy \,\propto \delta_{r\,s}. \tag{4.18}$$

It should be stressed that $P_r^{\Delta_S}(y)$, a *monic* polynomial in y, is also a polynomial in the parameter α with *all integer coefficients*.

4.2.2 $\mathcal{R} = \mathbf{V}$ for D_r

As in the previous example, $\mathbf{V} = \{\pm\mathbf{e}_j \,|\, j = 1, \ldots, r\}$, we introduce ($y \equiv x^2$, (3.7))

$$P_r^{\mathbf{V}}(y) \equiv P_{D_r}^{\mathbf{V}}(x) = \prod_{j=1}^{r} \left(x^2 - \tilde{q}_j^2\right) = (-1)^r r! L_r^{(-1)}(y). \tag{4.19}$$

They are orthogonal to each other:

$$\int_0^\infty P_r^{\mathbf{V}}(y) P_s^{\mathbf{V}}(y)\, y^{-1} e^{-y}\, dy \;\propto \int_0^\infty L_{r-1}^{(1)}(y) L_{s-1}^{(1)}(y) y\, e^{-y}\, dy \;\propto \delta_{r\,s}, \tag{4.20}$$

in which the identity (4.16) is used. Corresponding to the above mentioned Dynkin diagram folding $D_{r+1} \to B_r$ and (4.16), we obtain

$$x^2 P_{B_r}^{\Delta_S}(2|x) = P_{D_{r+1}}^{\mathbf{V}}(x) = P_{B_{r+1}}^{\Delta_S}(0|x). \tag{4.21}$$

4.2.3 $A_{2r-1} \to C_r$ and the relationship between Hermite and Laguerre polynomials

As is well-known the Dynkin diagram folding $A_{2r-1} \to C_r$ relates the A_{2r-1} Calogero system to the C_r (B_r) system with $\omega \to 2\omega$, $g_S(g_L) = 2g$ and $g_L(g_S) = g$, that is $\alpha = -1/2$. This would imply $P_{A_{2r-1}}^{\mathbf{V}}(x)$ (4.6) is equal to $P_{B_r}^{\Delta_S}(1/2|x)$ (4.17):

$$P_{A_{2r-1}}^{\mathbf{V}}(x) = P_{B_r}^{\Delta_S}(1/2|x), \tag{4.22}$$

which is equivalent to a well-known formula relating Hermite polynomials and Laguerre polynomials (eq(5.6.1) of [8]):

$$H_{2r}(x) = (-1)^r 2^{2r} r! L_r^{(-1/2)}(x^2), \quad H_{2r+1}(x) = (-1)^r 2^{2r+1} r! x L_r^{(1/2)}(x^2). \tag{4.23}$$

The former corresponds to $k = 1/2$ and (4.22). The latter corresponds to $k = 3/2$ and implies

$$P_{A_{2r}}^{\mathbf{V}}(x) = x P_{B_r}^{\Delta_S}(3/2|x). \tag{4.24}$$

Let us recall the corresponding results in the trigonometric case [8, 1]. The polynomial $P_{BC_r,\,c2}^{\Delta_{S+}}(k_1,\, k_2|x)$ (5.20) is proportional to Jacobi polynomial $P_r^{(\alpha,\beta)}(x)$ with $\alpha = k_1 + k_2 - 1$ and $\beta = k_2 - 1$. For $k_1 = 0$, $k_2 = 1/2$ ($k_1 = 0$, $k_2 = 3/2$) it reduces to the Chebyshev polynomial of the first (second) kind. As above, $k_1 = 0$, $k_2 = 1/2$ corresponds to the $A_{2r-1} \to C_r$ folding.

4.2.4 $\mathcal{R} = \mathbf{S}$ and $\bar{\mathbf{S}}$ for D_r

The spinor $\mathbf{S}$ and conjugate spinor $\bar{\mathbf{S}}$ representations of D_r are minimal representations with $D = 2^{r-1}$ and the natural normalisation $\mu^2 = r/4$. For odd r, we have the equality $-\mathbf{S} = \bar{\mathbf{S}}$ which means $P^{\mathbf{S}}_{D_r}(x) = P^{\bar{\mathbf{S}}}_{D_r}(x)$ for odd r. In fact, the symmetry of the D_r Dynkin diagram implies that the same formula holds for even r, too. Here we present $P^{\mathbf{S}}_{D_r}(x)$ for lower members of r:

$$
\begin{aligned}
P^{\mathbf{S},\bar{\mathbf{S}},\mathbf{V}}_{D_4}(x) &= x^2(-24 + 36x^2 - 12x^4 + x^6), && (4.25)\\
P^{\mathbf{S},\bar{\mathbf{S}}}_{D_5}(x) &= 25 - 3400x^2 + 13900x^4 - 20200x^6 + 12730x^8 - 3880x^{10}\\
&\quad +580x^{12} - 40x^{14} + x^{16}, && (4.26)\\
P^{\mathbf{S},\bar{\mathbf{S}}}_{D_6}(x) &= 2^{-16}\big(951356390625 - 24582413628000x^2 + 229552540380000x^4\\
&\quad -1001859665040000x^6 + 2271780895320000x^8 - 2992279237056000x^{10}\\
&\quad +2465846485977600x^{12} - 1332743493888000x^{14} + 486926396352000x^{16}\\
&\quad -122431951872000x^{18} + 21351239884800x^{20} - 2577889198080x^{22}\\
&\quad +212745830400x^{24} - 11668684800x^{26} + 403046400x^{28} - 7864320x^{30}\\
&\quad +65536x^{32}\big)\,. && (4.27)
\end{aligned}
$$

The equality of the three polynomials for $\mathbf{V}$, $\mathbf{S}$ and $\bar{\mathbf{S}}$ in D_4, (4.25) reflects the three fold symmetry of the D_4 Dynkin diagram.

4.2.5 $\mathcal{R} = \Delta_L$ for B_r and D_r

The set of long roots of B_r is $\Delta_L = \{\pm(\mathbf{e}_j - \mathbf{e}_l), \pm(\mathbf{e}_j + \mathbf{e}_l)\,|\,1 \le j < l \le r\}$. The polynomial $P^{\Delta_L}_{B_r}(k|x)$ can be expressed neatly in terms of the coefficients of the polynomial $P^{\Delta_S}_{B_r}(k|x)$ (4.17). Suppose we have two polynomials in y:

$$
f = \prod_{i=1}^{n}(y - x_i^2) = \sum_{i=0}^{n}(-1)^i a_i\, y^{n-i}, \tag{4.28}
$$

$$
g = \prod_{1\le i<j\le n}\Big(y - (x_i - x_j)^2\Big)\Big(y - (x_i + x_j)^2\Big). \tag{4.29}
$$

Let us denote $b_i = x_i^2$, then we obtain g as a symmetric polynomial in b_i:

$$
g = \prod_{1\le i<j\le n}\Big(y^2 - 2(b_i + b_j)y + (b_i - b_j)^2\Big), \tag{4.30}
$$

and $\{a_i\}$ are the basis of the symmetric polynomials in b_i:

$$a_i = \sum_{1\le j_1<\cdots<j_i\le n} b_{j_1}\cdots b_{j_i}. \tag{4.31}$$

Thus g can be expressed in terms of the coefficients $\{a_i\}$ of f with integer coefficients. For example:

$$\begin{aligned}
n=2: &\quad g = y^2 - 2a_1 y + a_1^2 - 4a_2, && (4.32)\\
n=3: &\quad g = y^6 - 4a_1y^5 + 2(3a_1^2 - a_2)y^4 - 2(2a_1^3 - a_1a_2 - 13a_3)y^3 \\
&\qquad + (a_1^4 + 2a_1^2a_2 - 7a_2^2 - 24a_1a_3)y^2 - 2(a_1^2 - 3a_2)(a_1a_2 - 9a_3)y \\
&\qquad + a_1^2a_2^2 - 4a_2^3 - 4a_1^3a_3 + 18a_1a_2a_3 - 27a_3^2. && (4.33)
\end{aligned}$$

If f is of rational coefficients, so is g.

We list $P^{\Delta_L}_{B_r}(k|x)$ for lower members of r. This includes $P^{\Delta}_{D_r}(x)$ as a special case of $k=0$. As remarked before, they are presented as polynomials in $y \equiv x^2$:

$$\begin{aligned}
P^{\Delta_L}_{B_2}(k|x) &= 4(1+k) - 4(1+k)y + y^2, && (4.34)\\
P^{\Delta_L}_{B_3}(k|x) &= 108\,(1+k)\,(2+k)^2 - 324\,(1+k)\,(2+k)^2\,y + 9\,(2+k)^2\,(41+32\,k)\,y^2 \\
&\quad -4\,(2+k)\left(99+88\,k+16\,k^2\right)\,y^3 + 6\,(2+k)\,(17+8\,k)\,y^4 \\
&\quad -12\,(2+k)\,y^5 + y^6, && (4.35)\\
P^{\Delta_L}_{B_4}(k|x) &= 27648\,(1+k)\,(2+k)^2(3+k)^3 - 165888\,(1+k)\,(2+k)^2(3+k)^3\,y \\
&\quad +4608(2+k)^2(3+k)^3\,(91+82\,k)\,y^2 \\
&\quad -512\,(2+k)\,(3+k)^3\left(2282+2777\,k+792\,k^2\right)y^3 \\
&\quad +192\,(2+k)\,(3+k)^2\left(15462+20235\,k+8336\,k^2+1088\,k^3\right)y^4 \\
&\quad -768\,(2+k)\,(3+k)^2\left(2085+2167\,k+688\,k^2+64\,k^3\right)y^5 \\
&\quad +64(3+k)^2\left(17634+22113\,k+9480\,k^2+1536\,k^3+64\,k^4\right)y^6 \\
&\quad -768(3+k)^2\left(342+327\,k+96\,k^2+8\,k^3\right)y^7 \\
&\quad +48\,(3+k)\left(2514+2465\,k+784\,k^2+80\,k^3\right)y^8 && (4.36)\\
&\quad -64\,(3+k)\left(186+123\,k+20\,k^2\right)y^9 + 240(3+k)^2y^{10} - 24\,(3+k)\,y^{11} + y^{12}.
\end{aligned}$$

As remarked above, $P^{\Delta_L}_{B_r}(k|x)$ is a polynomial in y and in k with *all integer coefficients* and is *monic* in y. The explicit forms of the polynomials $P^{\Delta_L}_{B_r}(k|x)$ $(r=5,6)$ and $P^{\Delta}_{D_r}(x)$

($r = 4,5,6$) can be found in [9]. The Dynkin diagram folding $D_{r+1} \to B_r$ relates the polynomials

$$P^{\Delta_L}_{B_r}(2|x)\left(P^{\Delta_S}_{B_r}(2|x)\right)^2 = P^{\Delta}_{D_{r+1}}(x) = P^{\Delta_L}_{B_{r+1}}(0|x), \tag{4.37}$$

which is the root version of the identity (4.21).

Next we discuss the systems based on the exceptional root systems. For these we have relied on numerical evaluation of the equilibrium points by Mathematica. Large enough digits of precision is maintained in internal computations, *e.g.*, we keep 2048 digits for E_8 Sutherland system. We have verified in each case that the fit of the polynomial with rational coefficients gives no detectable errors within the working precision.

4.3 E_r

The E series of the root systems, E_6, E_7 and E_8 are simply laced. The corresponding polynomials do not contain any coupling constants.

4.3.1 $\mathcal{R} = \mathbf{27}$ and Δ for E_6

Polynomials for **27** and Δ,

$$P^{\mathbf{27}}_{E_6}(x) = \prod_{\mu\in\mathbf{27}}(x - \mu\cdot\tilde{q}) \quad (\mu^2 = 4/3,\ \rho^2 = 2), \tag{4.38}$$

$$P^{\Delta}_{E_6}(x) = \prod_{\rho\in\Delta}(x - \rho\cdot\tilde{q}) \quad (\rho^2 = 2), \tag{4.39}$$

are slightly simplified for a different normalisation of $\mu \in \mathcal{R}$:

$$\begin{aligned} P^{\sqrt{1/3}\,\mathbf{27}}_{E_6}(x) &= 3^{-27/2}P^{\mathbf{27}}_{E_6}(\sqrt{3}x) = \prod_{\mu\in\mathbf{27}}(x - \hat{\mu}\cdot\tilde{q}) \quad (\hat{\mu} = \mu/\sqrt{3},\ \hat{\mu}^2 = 4/9) \\ &= x^3\,\big(200 - 3600\,x^2 + 24600\,x^4 - 83980\,x^6 + 162945\,x^8 - 192840\,x^{10} \\ &\qquad +144876\,x^{12} - 70416\,x^{14} + 22170\,x^{16} - 4440\,x^{18} + 540\,x^{20} - 36\,x^{22} + x^{24}\big), \end{aligned} \tag{4.40}$$

$$\begin{aligned} P^{\sqrt{1/3}\,\Delta}_{E_6}(x) &= 3^{-36}P^{\Delta}_{E_6}(\sqrt{3}x) = \prod_{\rho\in\Delta}(x - \hat{\rho}\cdot\tilde{q}) \quad (\hat{\rho} = \rho/\sqrt{3},\ \hat{\rho}^2 = 2/3) \\ &= \Big(81920 - 1474560\,x^2 + 8970240\,x^4 - 22749184\,x^6 + 28505088\,x^8 - 19829760\,x^{10} \\ &\qquad +8239872\,x^{12} - 2128896\,x^{14} + 346944\,x^{16} - 35328\,x^{18} + 2160\,x^{20} - 72\,x^{22} + x^{24}\Big) \\ &\quad \times\Big(200 - 3600\,x^2 + 24600\,x^4 - 83980\,x^6 + 162945\,x^8 - 192840\,x^{10} + 144876\,x^{12} \\ &\qquad -70416\,x^{14} + 22170\,x^{16} - 4440\,x^{18} + 540\,x^{20} - 36\,x^{22} + x^{24}\Big)^2 \end{aligned} \tag{4.41}$$

It is interesting to note that the second factor of $P^{\Delta}_{E_6}(x)$, (4.41), is the same as $P^{\mathbf{27}}_{E_6}(x)/x^3$, which is the same polynomial appearing in (4.40) and (4.47). Again it should be stressed that these polynomials are *monic* and all the coefficients are *integers*.

4.3.2 $\mathcal{R} = \mathbf{56}$ for E_7

Polynomial for **56**,

$$P^{\mathbf{56}}_{E_7}(x) = \prod_{\mu\in\mathbf{56}} (x - \mu\cdot\tilde{q}) \quad (\mu^2 = 3/2,\ \rho^2 = 2), \tag{4.42}$$

is slightly simplified for a different normalisation of μ:

$$\begin{aligned}
P^{\sqrt{2}\,\mathbf{56}}_{E_7}(x) &= 2^{28} P^{\mathbf{56}}_{E_7}(x/\sqrt{2}) = \prod_{\mu\in\mathbf{56}} (x - \hat{\mu}\cdot\tilde{q}) \quad (\hat{\mu} = \sqrt{2}\mu,\ \hat{\mu}^2 = 3) \\
&= 20441179226615503866613265625 - 4858344185249041690312528650 0\,x^2 \\
&\quad + 403943437764362721049483097250\,x^4 - 1594876299784237542505579618500\,x^6 \\
&\quad + 3423181532874686547792360316875\,x^8 - 4470973846715160163197028791000\,x^{10} \\
&\quad + 3844463042762881314328636794900x^{12} - 2298706753677324429083230164600x^{14} \\
&\quad + 994190889968661674517540390225\,x^{16} - 320292296385170629680242995500\,x^{18} \\
&\quad + 78600569652362205629789205150\,x^{20} - 14948636823173617875192068460\,x^{22} \\
&\quad + 2232949785098933644991402715\,x^{24} - 264680665744227895592493840\,x^{26} \\
&\quad + 25089285771398909108223000\,x^{28} - 1912398423761929885120080\,x^{30} \\
&\quad + 117632735062147883037411\,x^{32} - 5848529412061451267964\,x^{34} \\
&\quad + 234966118304680273854\,x^{36} - 7609794291104570460\,x^{38} \\
&\quad + 197734877929087065\,x^{40} - 4090765650038424\,x^{42} + 66612822142356\,x^{44} \\
&\quad - 839599815096\,x^{46} + 7991799795\,x^{48} - 55327860\,x^{50} \\
&\quad + 261954\,x^{52} - 756\,x^{54} + x^{56}
\end{aligned} \tag{4.43}$$

4.3.3 $\mathcal{R} = \Delta$ for E_7 and E_8

The polynomials $P^{\Delta}_{E_7}(x)$ and $P^{\Delta}_{E_8}(x)$ are too long to be displayed here. See [9] for explicit forms. It should be stressed that five *monic* polynomials in x, $P^{\sqrt{1/3}\,\mathbf{27}}_{E_6}(x)$ (4.40) (and $P^{\mathbf{27}}_{E_6}(x)$ (4.38)), $P^{\Delta}_{E_6}(x)$ (4.41), $P^{\sqrt{2}\,\mathbf{56}}_{E_7}(x)$ (4.43), $P^{\Delta}_{E_7}(x)$ and $P^{\Delta}_{E_8}(x)$ have *integer coefficients only*.

4.4 F_4

The theory has two coupling constants g_L and g_S for the long ($\rho_L^2 = 2$) and short ($\rho_S^2 = 1$) roots. We present the polynomials as a function of $k \equiv g_S/g_L$.

4.4.1 $\mathcal{R} = \Delta_L$ for F_4

$$
\begin{aligned}
P_4^L(k|y) \;\equiv\; & P_{F_4}^{\Delta_L}(k|x) = \prod_{\rho\in\Delta_L}(x-\rho\cdot\tilde{q}) = \prod_{\rho\in\Delta_{L+}}\left(y-(\rho\cdot\tilde{q})^2\right) \quad (\rho_L^2=2)\\
= \; & 746496(1+k)^6(2+k)^2(1+2k) - 4478976(1+k)^6(2+k)^2(1+2k)y\\
& +124416(1+k)^5(2+k)^2(1+2k)(91+64\,k)y^2\\
& -13824(1+k)^5(2+k)\left(2282+6049k+3712\,k^2+512\,k^3\right)y^3\\
& +15552(1+k)^4(2+k)\left(1718+5027k+4288\,k^2+1024\,k^3\right)y^4\\
& -20736(1+k)^4(2+k)\left(695+1472k+704\,k^2\right)y^5\\
& +1728(1+k)^3\left(5878+16235k+14408k^2+4096\,k^3\right)y^6\\
& -62208(1+k)^3(38+71k+32k^2)y^7 + 432(1+k)^2(838+1627k+784k^2)y^8\\
& -576(1+k)^2(62+61k)y^9 + 2160(1+k)^2y^{10} - 72(1+k)y^{11} + y^{12}. \qquad (4.44)
\end{aligned}
$$

4.4.2 $\mathcal{R} = \Delta_S$ for F_4

$$
\begin{aligned}
P_4^S(k|y) \;\equiv\; & P_{F_4}^{\Delta_S}(k|x) = \prod_{\rho\in\Delta_S}(x-\rho\cdot\tilde{q}) = \prod_{\rho\in\Delta_{S+}}\left(y-(\rho\cdot\tilde{q})^2\right) \quad (\rho_S^2=1)\\
= \; & 729k^3(1+k)^6(2+k)(1+2k)^2/4 - 2187k^2(1+k)^6(2+k)(1+2k)^2y\\
& +243k(1+k)^5(2+k)(1+2k)^2(64+91k)/2\,y^2\\
& -27(1+k)^5(1+2k)\left(512+3712k+6049k^2+2282k^3\right)y^3\\
& +243(1+k)^4(1+2k)(1024+4288k+5027k^2+1718k^3)/4\,y^4\\
& -162(1+k)^4(1+2k)\left(704+1472k+695k^2\right)y^5\\
& +27(1+k)^3\left(4096+14408k+16235k^2+5878k^3\right)y^6\\
& -1944(1+k)^3\left(32+71k+38k^2\right)y^7 + 27(1+k)^2\left(784+1627k+838k^2\right)y^8\\
& -72(1+k)^2(61+62k)y^9 + 540(1+k)^2y^{10} - 36(1+k)y^{11} + y^{12}. \qquad (4.45)
\end{aligned}
$$

They are related with each other reflecting the self-duality of the F_4 root system. If one replaces k by $1/k$ and y by $y/(2k)$ in $P_4^S(k|y)$, one obtains $P_4^L(k|y)/(2k)^{12}$:

$$
P_4^L(k|y) = (2k)^{12}P_4^S(1/k|y/2k), \quad \text{or} \quad P_{F_4}^{\Delta_L}(k|x) = (2k)^{12}P_{F_4}^{\Delta_S}(1/k|x/\sqrt{2k}). \qquad (4.46)
$$

It is well-known that F_4 with the coupling ratio $k = g_S/g_L = 2$ is obtained from E_6 by folding. This relates F_4 polynomials to E_6 polynomials:

$$P_{F_4}^{\Delta_S}(2|x) = P_{E_6}^{\mathbf{27}}(x)/x^3, \quad P_{F_4}^{\Delta_L}(2|x)\left(P_{F_4}^{\Delta_S}(2|x)\right)^2 = P_{E_6}^{\Delta}(x). \tag{4.47}$$

Both of them have trigonometric counterparts as will be shown later (5.64)-(5.66). The two polynomials $P_{F_4}^{\Delta_L}(k|x)$ and $P_{F_4}^{\sqrt{2}\Delta_S}(k|x)$ have *integer coefficients only*. This property seems to be inherited from E_6, too.

4.5 G_2

The theory has two coupling constants g_L and g_S for the long ($\rho_L^2 = 2$) and short ($\rho_S^2 = 2/3$) roots. We present the polynomials as a function of $k \equiv g_S/g_L$.

4.5.1 $\mathcal{R} = \Delta_L$ for G_2

$$\begin{aligned} P_2^L(k|y) \;&\equiv\; P_{G_2}^{\Delta_L}(k|x) = \prod_{\rho\in\Delta_L}(x - \rho\cdot\tilde{q}) = \prod_{\rho\in\Delta_{L+}}\left(y - (\rho\cdot\tilde{q})^2\right) \quad (\rho_L^2 = 2) \\ &= \; -27(1+k)^2/2 + 81(1+k)^2/4\,y - 9(1+k)y^2 + y^3. \end{aligned} \tag{4.48}$$

4.5.2 $\mathcal{R} = \Delta_S$ for G_2

$$\begin{aligned} P_2^S(k|y) \;&\equiv\; P_{G_2}^{\Delta_S}(k|x) = \prod_{\rho\in\Delta_S}(x - \rho\cdot\tilde{q}) = \prod_{\rho\in\Delta_{S+}}\left(y - (\rho\cdot\tilde{q})^2\right) \quad (\rho_S^2 = 2/3) \\ &= \; -k(1+k)^2/2 + 9(1+k)^2/4\,y - 3(1+k)y^2 + y^3. \end{aligned} \tag{4.49}$$

They are related with each other reflecting the self-duality of the G_2 root system:

$$P_2^L(k|y) = (3k)^3 P_2^S(1/k|y/3k), \quad \text{or} \quad P_{G_2}^{\Delta_L}(k|x) = (3k)^3 P_{G_2}^{\Delta_S}(1/k|x/\sqrt{3k}). \tag{4.50}$$

The G_2 Calogero system with the coupling ratio $k = g_S/g_L = 3$ is obtained from that of D_4 by the three-fold folding $D_4 \to G_2$. This implies analogous relations to (4.47)

$$P_{G_2}^{\Delta_S}(3|x) = P_{D_4}^{\mathcal{R}}(x)/x^2 \quad (\mathcal{R} = \mathbf{V}, \mathbf{S}, \bar{\mathbf{S}}), \quad P_{G_2}^{\Delta_L}(3|x)\left(P_{G_2}^{\Delta_S}(3|x)\right)^3 = P_{D_4}^{\Delta}(x). \tag{4.51}$$

Both of them have trigonometric counterparts, too, as will be shown later. The two polynomials $P_{G_2}^{\sqrt{2}\Delta_L}(k|x)$ and $P_{G_2}^{\sqrt{2}\Delta_S}(k|x)$ have *integer coefficients only*. This property seems to be inherited from D_4.

Thirdly let us discuss the systems based on non-crystallographic root systems.

4.6 $I_2(m)$

The equilibrium points are easily obtained when parametrised by the two-dimensional polar coordinates [1]:

$$\bar{q} = (\bar{q}_1, \bar{q}_2) = \bar{r}(\sin\bar{\varphi}, \cos\bar{\varphi}), \tag{4.52}$$

$$\bar{r}^2 = \frac{mg}{\omega}, \ \ \bar{\varphi} = \frac{\pi}{2m} \ \ (m: \ \text{odd}) \ ; \quad \bar{r}^2 = \frac{m(g_e+g_o)}{2\omega}, \ \ \tan\frac{m\bar{\varphi}}{2} = \sqrt{\frac{g_e}{g_o}} \ \ (m: \ \text{even}), \tag{4.53}$$

in which g is the coupling constant in the simply laced odd m theory, whereas g_o (g_e) is the coupling constant for odd (even) roots in the non-simply laced even m theory. As $\mathcal{R}$ we choose the set of the vertices of the regular m-gon R_m on which the dihedral group $I_2(m)$ acts:

$$R_m = \Big\{ (\cos(2j\pi/m + t_0), \sin(2j\pi/m + t_0)) \in \mathbb{R}^2 \ \Big| \ j = 1, \ldots, m \Big\}, \tag{4.54}$$
$$t_0 = \pi/2m \ \ (m: \ \text{odd}) \ ; \quad t_0 = 0 \ \ (m: \ \text{even}).$$

The polynomial $\prod_{\mu\in R_m}(x - \mu\cdot\tilde{q})$ (3.2) is obtained trivially:

$$P_m(x) \equiv P^{R_m}_{I_2(m)}(x) = \prod_{\mu\in R_m}(x - \mu\cdot\tilde{q}) = \prod_{j=1}^{m}\left(x - \sin\Big(\frac{2j\pi}{m} + \frac{\varphi_0}{m}\Big)\right), \tag{4.55}$$

in which

$$\varphi_0 = \pi \ \ (m: \ \text{odd}) \ ; \quad \varphi_0 = 2\arctan\sqrt{k}, \ \ k \equiv g_e/g_o \ \ (m: \ \text{even}). \tag{4.56}$$

For odd m $P_m(x)$ is proportional to the Chebyshev polynomial of the first kind $T_m(x)$ (see (5.4)). For even m and for the equal coupling $g_e = g_o$, $P_m(x)$ is also proportional to the Chebyshev polynomial $T_m(x)$ and thus the entire $\{P_m(x) = 2^{1-m}T_m(x)\}$ constitute orthogonal polynomials [1]. For generic coupling $g_e \neq g_o$ the orthogonality no longer holds. This can be seen most easily by the explicit forms of the lower members of P_{even} in the non-singular limiting cases, $g_e = 0$ and $g_o = 0$:

$$\begin{aligned} g_e = 0 \ &: \quad x^2, \ x^2(x^2-1), \ x^2(x^2-3/4)^2, \ x^2(x^2-1/2)^2(x^2-1), \ \ldots, \\ g_o = 0 \ &: \quad x^2-1, \ (x^2-1/2)^2, \ (x^2-1)(x^2-1/4)^2, \ (x^4-x^2+1/8)^2, \ \ldots, \end{aligned} \tag{4.57}$$

which have definite sign in $-1 < x < 1$.

The following equivalences are well-known: $A_2 \equiv I_2(3)$, $B_2 \equiv I_2(4)$ and $G_2 \equiv I_2(6)$. The $I_2(3)$ polynomial corresponds to the A_2 polynomial of vector $\mathbf{V}$,

$$P^{R_3}_{I_2(3)}(x) = \frac{1}{4}T_3(x) = \frac{1}{16\sqrt{2}}H_3(\sqrt{2}x) = P^{\mathbf{V}/\sqrt{2}}_{A_2}(x). \tag{4.58}$$

As for $I_2(4)$ polynomial, we obtain from (4.55)

$$P^{R_4}_{I_2(4)}(x) = x^4 - x^2 + \frac{k}{4(1+k)}, \quad k \equiv g_e/g_o. \tag{4.59}$$

For the B_2 system, the Laguerre polynomial with $\alpha = k - 1 \equiv g_e/g_o - 1$ reads

$$L^{(\alpha)}_2(y) = \frac{1}{2}y^2 - (k+1)y + k(1+k)/2, \quad \alpha = k - 1.$$

They are proportional to each other upon identification $y = 2(1+k)x^2$. The $I_2(6)$ polynomial obtained from (4.55) reads, after some calculation

$$P^{R_6}_{I_2(6)}(x) = x^6 - \frac{3}{2}x^4 + \frac{9}{16}x^2 - \frac{k}{16(1+k)}, \quad k = g_e/g_o, \tag{4.60}$$

which is proportional to $P^S_2(k|y)$ (4.49) upon the same identification as above $y = 2(1+k)x^2$.

4.7 H_3 and H_4

The non-crystallographic H_3 and H_4 are simply laced root systems. In both cases the roots are normalised to 2, as with the other simply laced root systems, $\rho^2 = 2$. Then both monic polynomials $P^{\Delta}_{H_3}(x)$ and $P^{\Delta}_{H_4}(x)$ have *integer coefficients only*.

4.7.1 $\mathcal{R} = \Delta$ for H_3

$$\begin{aligned} P^{\Delta}_3(y) &\equiv P^{\Delta}_{H_3}(x) = \prod_{\rho\in\Delta}(x - \rho\cdot\tilde{q}) = \prod_{\rho\in\Delta_+}\left(y - (\rho\cdot\tilde{q})^2\right) \quad (\rho^2 = 2) \\ &= \left(-450 + 225\,y - 30\,y^2 + y^3\right) \\ &\quad\times\left(5625 - 22500\,y + 27000\,y^2 - 9600\,y^3 + 1200\,y^4 - 60\,y^5 + y^6\right) \\ &\quad\times\left(22500 - 67500\,y + 46125\,y^2 - 11700\,y^3 + 1275\,y^4 - 60\,y^5 + y^6\right). \end{aligned} \tag{4.61}$$

4.7.2 $\mathcal{R} = \Delta$ for H_4

$$P^{\Delta}_4(y) \equiv P^{\Delta}_{H_4}(x) = \prod_{\rho\in\Delta}(x - \rho\cdot\tilde{q}) = \prod_{\rho\in\Delta_+}\left(y - (\rho\cdot\tilde{q})^2\right) \quad (\rho^2 = 2)$$

$$= (656100000000 - 1093500000000\,y + 601425000000\,y^2 - 154305000000\,y^3 + 21343500000\,y^4$$

$$-1701000000\,y^5 + 80392500\,y^6 - 2250000\,y^7 + 36000\,y^8 - 300\,y^9 + y^{10})$$

$$\times(74733890625000000000 - 99645187500000000000\,y + 45172485000000000000\,y^2$$

$$-9092623359375000000\,y^3 + 929285482781250000000\,y^4$$

$$
\begin{aligned}
&-5284191674250000000000\,y^5+18358385767875000000000\,y^6\\
&-41697451350000000000\,y^7+6488441285906250000000\,y^8-71483472810000000000\,y^9\\
&+570711449970000000\,y^{10}-3355802966250000000\,y^{11}+146832674062500000\,y^{12}\\
&-480384270000000\,y^{13}+11739694500000\,y^{14}-212600700000\,y^{15}+2804085000\,y^{16}\\
&-26100000\,y^{17}+162000\,y^{18}-600\,y^{19}+y^{20})\\
\times&(13620251566406250000000000000000-20430377349609375000000000000000\,y\\
&+11066454397705078125000000000000\,y^2-28067252402893359375000000000000\,y^3\\
&+40655043499727460937500000000000\,y^4-37708990350047957812500000000000\,y^5\\
&+24038577591496497070312500000000\,y^6-11046797751535192968750000000000\,y^7\\
&+3787974011029930593750000000000\,y^8-994761504511959206250000000000\,y^9\\
&+20412896044083175420312500000000\,y^{10}-3325061946787265812500000000000\,y^{11}\\
&+435293400958687490625000000000\,y^{12}-462459240055472934375000000000\,y^{13}\\
&+40174637528621421562500000000\,y^{14}-2870118137602987875000000000\,y^{15}\\
&+169317335092151475000000000\,y^{16}-8269924499168062500000000\,y^{17}\\
&+334831824489389062500000\,y^{18}-11234993640754500000000\,y^{19}\\
&+311856586899345000000\,y^{20}-7135295128312500000\,y^{21}+133798076606250000\,y^{22}\\
&-2038887247500000\,y^{23}+24945262200000\,y^{24}-240849450000\,y^{25}+1789742250\,y^{26}\\
&-98550000\,y^{27}+378000\,y^{28}-900\,y^{29}+y^{30}).
\end{aligned}
\tag{4.62}
$$

5 Sutherland Systems

Let us first discuss the systems based on the classical root systems.

5.1 A_r

The equilibrium position is "*equally-spaced*" (see eq.(5.14) of [1]) and translational invariant. We choose the constant shift such that the coordinate of "center of mass" vanishes, $\sum_{j=1}^{r+1}\bar{q}_j=0$:

$$
\bar{q}_j=\frac{\pi(r+1-j)}{r+1}-\frac{\pi r}{2(r+1)}=\frac{\pi}{2}-\frac{\pi(2j-1)}{2(r+1)}=-\bar{q}_{r+2-j}\quad(j=1,\ldots,r+1).
\tag{5.1}
$$

5.1.1 $\mathcal{R} = \mathbf{V}$ for A_r

For the vector weight $\mu_j \in \mathbf{V}$ (4.5), $\mu_j \cdot \bar{q}$ is independent on the constant shift of $\bar{q}$. The above choice (5.1) leads to

$$\mu_j \cdot \bar{q} = \frac{\pi}{2} - \frac{\pi(2j-1)}{2(r+1)} = q_j, \quad -\frac{\pi}{2} < \mu_j \cdot \bar{q} < \frac{\pi}{2} \quad (j = 1, \ldots, r+1). \tag{5.2}$$

In this case the polynomial (3.4) is given by

$$P_r^{\mathbf{V}}(x) \equiv P_{A_r,\,s}^{\mathbf{V}}(x) = \prod_{j=1}^{r+1}\Big(x - \sin(\mu_j \cdot \bar{q})\Big) = \prod_{j=1}^{r+1}\left(x - \cos\frac{\pi(2j-1)}{2(r+1)}\right) = 2^{-r}T_{r+1}(x)\,. \tag{5.3}$$

Here $T_n(\cos\varphi) = \cos(n\varphi)$ is the Chebyshev polynomial of the first kind, whose Rodrigues' formula is

$$T_n(x) = \frac{(-1)^n}{(2n-1)!!}(1-x^2)^{1/2}\left(\frac{d}{dx}\right)^n (1-x^2)^{n-1/2} = 2^{n-1}x^n + \cdots. \tag{5.4}$$

They are orthogonal to each other:

$$\int_{-1}^{1} \frac{P_r^{\mathbf{V}}(x)P_s^{\mathbf{V}}(x)}{\sqrt{1-x^2}}\,dx \,\propto \delta_{r\,s}. \tag{5.5}$$

This is a new result. Another definition

$$P_{A_r,\,s}^{\mathbf{V}\,\prime}(x) = \prod_{j=1}^{r+1}\Big(x - 2\sin(\mu_j \cdot \bar{q})\Big) = 2T_{r+1}(x/2) = 2^{r+1}P_{A_r,\,s}^{\mathbf{V}}(x/2) \tag{5.6}$$

provides a *monic* polynomial with *all integer coefficients*.

It is easy to see that

$$P_{A_r,\,c}^{\mathbf{V}}(x) = \prod_{j=1}^{r+1}\Big(x - \cos(\mu_j \cdot \bar{q})\Big) = \prod_{j=1}^{r+1}\left(x - \sin\frac{\pi(2j-1)}{2(r+1)}\right)$$

does not give rational polynomials, for example, $P_{A_1,\,c}^{\mathbf{V}}(x) = x^2 - \sqrt{2}x + 1/2$. In fact, in most cases the polynomial $P_{\Delta,\,c}^{\mathcal{R}}(x)$ is not of rational coefficients. In the rest of this paper we will not consider this type of polynomials.

The other polynomials,

$$P_{A_r,\,s2}^{\mathbf{V}}(x) = \prod_{j=1}^{r+1}\Big(x - \sin(2\mu_j \cdot \bar{q})\Big) = \prod_{j=1}^{r+1}\left(x - \sin\frac{\pi(2j-1)}{r+1}\right),$$

$$P_{A_r,\,c2}^{\mathbf{V}}(x) = \prod_{j=1}^{r+1}\Big(x - \cos(2\mu_j \cdot \bar{q})\Big) = \prod_{j=1}^{r+1}\left(x + \cos\frac{\pi(2j-1)}{r+1}\right),$$

are essentially the same as $P^{\mathbf{V}}_{A_r,\,s}(x)$, (5.3). Only the constant term can be different:

$$P^{\mathbf{V}}_{A_r,\,s2}(x)-P^{\mathbf{V}}_{A_r,\,s}(x)=-2^{-r}\sin\frac{\pi r}{2},\quad P^{\mathbf{V}}_{A_r,\,c2}(x)-P^{\mathbf{V}}_{A_r,\,s}(x)=(-1)^{r+1}2^{-r}. \tag{5.7}$$

Thus we consider only the polynomial $P^{\mathcal{R}}_{A_r,\,s}(x)=\prod_{\mu\in\mathcal{R}}\Big(x-\sin(\mu\cdot\bar q)\Big)$ for various $\mathcal{R}$ of A_r.

5.1.2 $\mathcal{R}=\mathbf{V}_i$ for A_r

From (4.9) and (5.2), the polynomial (3.4) is given by

$$P^{\mathbf{V}_i}_{A_r,\,s}(x)=\prod_{1\le j_1<\cdots<j_i\le r+1}\Big(x-\sin(\bar q_{j_1}+\cdots\bar q_{j_i})\Big)=P^{\mathbf{V}_{r+1-i}}_{A_r,\,s}(x). \tag{5.8}$$

This polynomial can be expressed in terms of the coefficients of $T_{r+1}(x)$ by the same method as given in section 5.2.5, and $2^{D_i}P^{\mathbf{V}_i}_{A_r,s}(x/2)$ gives a monic polynomial with integer coefficients. See [9] for the explicit forms of the polynomials $P^{\mathbf{V}_i}_{A_r,\,s}(x)$ of lower rank r.

As in the Calogero case, we report only on $\mathbf{V}_2$:

$$\begin{aligned}P^{\mathbf{V}_2}_{A_r,\,s}(x)&=\prod_{1\le j<l\le r+1}\Big(x-\sin(\bar q_j+\bar q_l)\Big) \qquad (5.9)\\ &=\begin{cases} x^{(r+1)/2}\displaystyle\prod_{1\le j<l\le (r+1)/2}\Big(x^2-\sin^2(\bar q_j-\bar q_l)\Big)\Big(x^2-\sin^2(\bar q_j+\bar q_l)\Big) & r\,:\,\text{odd}\\ x^{r/2}\displaystyle\prod_{j=1}^{r/2}(x^2-\sin^2\bar q_j)\cdot\prod_{1\le j<l\le r/2}\Big(x^2-\sin^2(\bar q_j-\bar q_l)\Big)\Big(x^2-\sin^2(\bar q_j+\bar q_l)\Big) & r\,:\,\text{even.}\end{cases}\end{aligned}$$

Based on the fact that the zeros of Chebyshev and Jacobi polynomials are related as seen from the formulae (5.28) and (5.29), this can be expressed by using the polynomials associated with the BC_r Sutherland systems in the following way:

$$P^{\mathbf{V}_2}_{A_{2r-1},\,s}(x)=2^{-r(r-1)}x^rP^{\Delta_{M+}}_{BC_r,\,c2}(0,1/2|1-2x^2), \tag{5.10}$$

$$P^{\mathbf{V}_2}_{A_{2r},\,s}(x)=2^{-r(r-1)}x^{r-1}P^{\mathbf{V}}_{A_{2r},\,s}(x)P^{\Delta_{M+}}_{BC_r,\,c2}(1,1/2|1-2x^2). \tag{5.11}$$

The explicit forms of the functions $P^{\Delta_{M+}}_{BC_r,\,c2}(k_1,k_2|x)$ for lower r are given in section 5.2.5.

5.1.3 $\mathcal{R}=\Delta$ for A_r

The polynomial has a factorized form:

$$P^{\Delta}_{A_r,\,s}(x)=\prod_{1\le j<l\le r+1}\Big(x^2-\sin^2(\bar q_j-\bar q_l)\Big)=\begin{cases}x^2-1 & (r=1)\\ x^{-r-1}P^{\mathbf{V}}_{A_r,\,s2}(x)\Big(P^{\mathbf{V}_2}_{A_r,\,s}(x)\Big)^2 & (r\ge 2)\,.\end{cases} \tag{5.12}$$

It is elementary to evaluate $P^{\Delta}_{A_r,\,s}(x)$ for lower rank:

$$
\begin{aligned}
P^{\Delta}_{A_r,\,s}(x) &= \prod_{1\le j<l<r+1}\left(x^2-\sin^2\!\left(\frac{\pi(l-j)}{r+1}\right)\right),\\
P^{\Delta}_{A_1,\,s}(x) &= x^2-1,\\
P^{\Delta}_{A_2,\,s}(x) &= 2^{-6}(4x^2-3)^3,\\
P^{\Delta}_{A_3,\,s}(x) &= 2^{-4}(x^2-1)^2(2x^2-1)^4,\\
P^{\Delta}_{A_4,\,s}(x) &= 2^{-20}(5-20x^2+16x^4)^5,\\
P^{\Delta}_{A_5,\,s}(x) &= 2^{-24}(x^2-1)^3(4x^2-1)^6(4x^2-3)^6,\\
P^{\Delta}_{A_6,\,s}(x) &= 2^{-42}(-7+56\,x^2-112\,x^4+64\,x^6)^7.
\end{aligned}
$$

For $r=1,3$ and 5, $P^{\Delta}_{A_r,\,s}(x)$ are of definite sign in $-1<x<1$. They can never be orthogonal with each other for whichever choice of the positive definite weight function.

5.2 BC_r and D_r

As shown in [1], the equations (2.7) for $\Delta = BC_r$ read

$$
-2g_M\sum_{\substack{l=1\\ l\neq j}}^{r}\frac{\sin 2\bar{q}_j}{\cos 2\bar{q}_j-\cos 2\bar{q}_l}+g_S\frac{\cos\bar{q}_j}{\sin\bar{q}_j}+2g_L\frac{\cos 2\bar{q}_j}{\sin 2\bar{q}_j}=0\quad (j=1,\ldots,r). \tag{5.13}
$$

For non-vanishing g_S and g_L, $\sin 2\bar{q}_j = 0$ cannot satisfy the above equation. Thus by dividing by $\sin 2\bar{q}_j$ we obtain for $k_1 \equiv g_S/g_M$, $k_2 \equiv g_L/g_M$:

$$
\sum_{\substack{l=1\\ l\neq j}}^{r}\frac{1}{\bar{x}_j-\bar{x}_l}+\frac{k_1+k_2}{2(\bar{x}_j-1)}+\frac{k_2}{2(\bar{x}_j+1)}=0\quad (j=1,\ldots,r), \tag{5.14}
$$

in which $\bar{x}_j \equiv \cos 2\bar{q}_j$. These are the equations satisfied by the zeros $\{\bar{x}_j\,|\,j=1,\ldots,r\}$ of the Jacobi polynomial $P^{(\alpha,\beta)}_r(x)$ [8] with

$$
\alpha = k_1+k_2-1,\quad \beta = k_2-1. \tag{5.15}
$$

The Rodrigues' formula for the Jacobi polynomial $P^{(\alpha,\beta)}_n(x)$ reads

$$
\begin{aligned}
P^{(\alpha,\beta)}_n(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right) \qquad (5.16)\\
&= \frac{1}{2^n n!}\frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}x^n+\cdots.
\end{aligned}
$$

For $\Delta = D_r$, we have $g_S = g_L = 0$, implying $\alpha = \beta = -1$. We choose

$$\bar{q}_1 = 0, \quad \bar{q}_r = \pi/2 \quad (\Longleftrightarrow \cos 2\bar{q}_1 = 1, \quad \cos 2\bar{q}_r = -1),$$

then (2.7) read

$$\sum_{\substack{l=2\\ l\neq j}}^{r-1} \frac{1}{\bar{x}_j - \bar{x}_l} + \frac{1}{\bar{x}_j - 1} + \frac{1}{\bar{x}_j + 1} = 0 \quad (j = 2, \ldots, r-1), \tag{5.17}$$

in which $\bar{x}_j \equiv \cos 2\bar{q}_j$ $(j = 2, \ldots, r-1)$. These are the equations satisfied by the zeros $\{\bar{x}_j \,|\, j = 2, \ldots, r-1\}$ of the Jacobi polynomial $P_{r-2}^{(1,1)}(x)$ [8]. In fact, there is an identity

$$4P_r^{(-1,-1)}(x) = (x^2 - 1) P_{r-2}^{(1,1)}(x), \tag{5.18}$$

which means that $\{1, \bar{x}_2, \ldots, \bar{x}_{r-1}, -1\}$ are the zeros of $P_r^{(-1,-1)}(x)$. This allows to treat D_r as a limiting case of BC_r.

The possible $\mathcal{R}$'s for BC_r are Δ_S, Δ_M and Δ_L. Since $\Delta_S = \{\pm \mathbf{e}_j \,|\, j = 1, \ldots, r\}$ and $\Delta_L = \{\pm 2\mathbf{e}_j \,|\, j = 1, \ldots, r\}$, we have trivial identities among the polynomials

$$P^{\Delta_L}_{BC_r,\,s}(k_1,\, k_2|x) = P^{\Delta_S}_{BC_r,\,s2}(k_1,\, k_2|x), \quad P^{\Delta_L}_{BC_r,\,c}(k_1,\, k_2|x) = P^{\Delta_S}_{BC_r,\,c2}(k_1,\, k_2|x). \tag{5.19}$$

In other words, these relations prompted us to introduce the polynomials of the forms $\prod_{\mu\in\mathcal{R}} (x - \sin(2\mu\cdot\bar{q}))$ and $\prod_{\mu\in\mathcal{R}} (x - \cos(2\mu\cdot\bar{q}))$. For BC_r Sutherland system we consider $\mathcal{R} = \Delta_S$ and Δ_M only.

5.2.1 $\mathcal{R} = \Delta_S$ for BC_r

Since Δ_S is *even* and that $\{\bar{x}_j = \cos 2\bar{q}_j \,|\, j = 1, \ldots, r\}$ are the zeros of the Jacobi polynomial, it is natural to consider the polynomial (3.8)

$$P^{\Delta_{S+}}_{BC_r,\,c2}(k_1, k_2|x) = \prod_{j=1}^{r} (x - \cos 2\bar{q}_j) = 2^r r! \, \frac{\Gamma(r + \alpha + \beta + 1)}{\Gamma(2r + \alpha + \beta + 1)} \, P_r^{(\alpha,\beta)}(x), \tag{5.20}$$

with $\alpha = k_1 + k_2 - 1$ and $\beta = k_2 - 1$. They are orthogonal to each other:

$$\int_{-1}^{1} P_r^{(\alpha,\beta)}(x) P_s^{(\alpha,\beta)}(x)\, (1-x)^\alpha (1+x)^\beta \, dx \, \propto \delta_{r\,s}. \tag{5.21}$$

As remarked in (3.9), the polynomial $P^{\Delta_{S+}}_{BC_r,\,c2}(k_1, k_2|x)$ is equivalent with $P^{\Delta_S}_{BC_r,\,s}(k_1, k_2|x)$. Needless to say that $2^n n! P_n^{(\alpha,\beta)}(x)$ is a polynomial in the parameters α and β with integer coefficients. Thus $P^{\Delta_{S+}}_{BC_r,\,c2}(k_1,k_2|x)$ (5.20) is a rational function in α and β with integer coefficients.

The other polynomial $P^{\Delta_S}_{BC_r,\,s2}(k_1,k_2|x)$ can be easily obtained by (3.10):

$$\begin{aligned} P^{\Delta_S}_{BC_r,\,s2}(k_1,k_2|x) &= \prod_{j=1}^{r}\Big(x^2-\sin^2(2\bar{q}_j)\Big) \\ &= (-1)^r\left(2^r r!\,\frac{\Gamma(r+\alpha+\beta+1)}{\Gamma(2r+\alpha+\beta+1)}\right)^2 P^{(\alpha,\beta)}_r(u)P^{(\beta,\alpha)}_r(u), \qquad (5.22) \end{aligned}$$

in which $u^2=1-x^2$. Remark that $P^{(\alpha,\beta)}_r(-x)=(-1)^r P^{(\beta,\alpha)}_r(x)$.

5.2.2 $\mathcal{R}=\mathbf{V}$ for D_r

This is a special ($k_1=k_2=0$ or $\alpha=\beta=-1$) case of the previous example. As in the previous example, we introduce

$$\begin{aligned} P^{\mathbf{V}_+}_r(x) \equiv P^{\mathbf{V}_+}_{D_r,\,c2}(x) &= \prod_{j=1}^{r}(x-\cos 2\bar{q}_j) = \frac{2^r r!(r-2)!}{(2r-2)!}P^{(-1,-1)}_r(x) \\ &= (x+1)(x-1)\prod_{j=2}^{r-1}(x-\bar{x}_j) = \frac{2^{r-2}r!(r-2)!}{(2r-2)!}(x+1)(x-1)P^{(1,1)}_{r-2}(x). \quad (5.23) \end{aligned}$$

They are orthogonal to each other:

$$\int_{-1}^{1} P^{\mathbf{V}_+}_r(x)P^{\mathbf{V}_+}_s(x)\,(1-x)^{-1}(1+x)^{-1}\,dx \,\propto \int_{-1}^{1} P^{(1,1)}_{r-2}(x)P^{(1,1)}_{s-2}(x)(1-x)(1+x)\,dx \,\propto \delta_{r\,s}. \qquad (5.24)$$

Corresponding to the Dynkin diagram folding $D_{r+1}\to B_r$ and (5.23), we obtain

$$(x-1)P^{\Delta_{S+}}_{BC_r,\,c2}(2,0|x) = P^{\mathbf{V}_+}_{D_{r+1},\,c2}(x) = P^{\Delta_{S+}}_{BC_{r+1},\,c2}(0,0|x), \qquad (5.25)$$

which is the trigonometric counterpart of (4.21).

The other polynomial $P^{\mathbf{V}}_{D_r,\,s2}(x)$ has a simple form

$$\begin{aligned} P^{\mathbf{V}}_{D_r,\,s2}(x) &= \prod_{j=1}^{r}\Big(x^2-\sin^2(2\bar{q}_j)\Big) \\ &= (-1)^r\left(\frac{2^{r-1}r!\,(r-2)!}{(2r-2)!}\right)^2 x^4\,\left(P^{(1,1)}_{r-2}(u)\right)^2\Big|_{u^2\to 1-x^2}, \qquad (5.26) \end{aligned}$$

which is of a definite sign in $-1<x<1$. Thus they do not form any orthogonal polynomials.

5.2.3 $A_{2r-1} \to C_r$ and the relationship between Chebyshev and Jacobi polynomials

As in the Calogero case, the Dynkin diagram folding $A_{2r-1} \to C_r$ implies

$$P^{\mathbf{V}}_{A_{2r-1},\,s}(x) = (-2)^{-r} P^{\Delta_{S+}}_{BC_r,\,c2}(0, 1/2|1-2x^2). \tag{5.27}$$

Indeed there are relations between Chebyshev and Jacobi polynomials:

$$\begin{aligned} 2^{1-2r} T_{2r}(x) &= (-1)^r \frac{r!(r-1)!}{(2r-1)!} P_r^{(-1/2,-1/2)}(1-2x^2), & (5.28)\\ 2^{-2r} T_{2r+1}(x) &= (-1)^r \frac{(r!)^2}{(2r)!} x P_r^{(1/2,-1/2)}(1-2x^2), & (5.29) \end{aligned}$$

on top of the well-known relations (eq(4.1.7) of [8]):

$$\frac{1\cdot 3\cdots(2r-1)}{2\cdot 4\cdots 2r} T_r(x) = P_r^{(-1/2,-1/2)}(x).$$

The former corresponds to (5.27) and the latter implies

$$P^{\mathbf{V}}_{A_{2r},\,s}(x) = (-2)^{-r} x P^{\Delta_{S+}}_{BC_r,\,c2}(1, 1/2|1-2x^2). \tag{5.30}$$

5.2.4 $\mathcal{R} = \mathbf{S}$ and $\bar{\mathbf{S}}$ for D_r

As in the Calogero systems, the symmetry of the D_r Dynkin diagram implies that $P^{\mathbf{S}}_{D_r,\,a}(x) = P^{\bar{\mathbf{S}}}_{D_r,\,a}(x)$, $a = s, c, s2, c2$. Among them $P^{\mathbf{S},\bar{\mathbf{S}}}_{D_r,\,c}(x)$ do not always give rational polynomials. As remarked above (3.9), $P^{\mathbf{S},\bar{\mathbf{S}}}_{D_r,\,s}(x)$ are equivalent to $P^{\mathbf{S}_+,\bar{\mathbf{S}}_+}_{D_r,\,c2}(x)$ for even rank r. Thus we list for lower rank r the polynomials $P^{\mathbf{S}_+,\bar{\mathbf{S}}_+}_{D_r,\,c2}(x)$ and $P^{\mathbf{S},\bar{\mathbf{S}}}_{D_r,\,s2}(x)$:

$$\begin{aligned} P^{\mathbf{V}_+,\mathbf{S}_+,\bar{\mathbf{S}}_+}_{D_4,\,c2}(x) &= (x^2-1)(x^2-1/5), & (5.31)\\ P^{\mathbf{V},\mathbf{S},\bar{\mathbf{S}}}_{D_4,\,s2}(x) &= x^4(x^2-4/5)^2, & (5.32)\\ P^{\mathbf{S},\bar{\mathbf{S}}}_{D_5,\,c2}(x) &= (x^2-1/2)^4(x^4-x^2-1/196)^2, & (5.33)\\ P^{\mathbf{S},\bar{\mathbf{S}}}_{D_5,\,s2}(x) &= (x^2-1/2)^4(x^4-x^2-1/196)^2, & (5.34)\\ P^{\mathbf{S}_+,\bar{\mathbf{S}}_+}_{D_6,\,c2}(x) &= 3^{-4}7^{-3}x^4(21x^4-28x^2+8)^2(63x^4-72x^2+16), & (5.35)\\ P^{\mathbf{S},\bar{\mathbf{S}}}_{D_6,\,s2}(x) &= 3^{-8}7^{-6}(x^2-1)^4(21x^4-14x^2+1)^4(63x^4-54x^2+7)^2. & (5.36) \end{aligned}$$

It is interesting to note that the formula (3.10) applies to D_5 (conjugate) spinor representation $\mathbf{S}$ ($\bar{\mathbf{S}}$), which is not *even*. This is because the set of values $\{\mu\cdot\bar{q}\,|\,\mu\in\mathbf{S}\}$ is *even*. Moreover, the function in (5.33) is invariant under $x^2 \to 1-x^2$.

5.2.5 $\mathcal{R} = \Delta_M$ for BC_r

The set of middle roots is $\Delta_M = \{\pm(\mathbf{e}_j - \mathbf{e}_l), \pm(\mathbf{e}_j + \mathbf{e}_l) \,|\, 1 \le j < l \le r\}$. As in the Calogero systems in 4.2.5, the polynomial $P^{\Delta_M}_{B_r,\,s}(k|x)$ can be expressed neatly in terms of the coefficients of the polynomial $P^{\Delta_S}_{B_r,\,s}(k|x)$. Suppose we have two polynomials in y:

$$f = \prod_{i=1}^{n}(y - \sin^2 x_i) = \sum_{i=0}^{n}(-1)^i a_i\, y^{n-i}, \tag{5.37}$$

$$g = \prod_{1\le i<j\le n}\Big(y - \sin^2(x_i - x_j)\Big)\Big(y - \sin^2(x_i + x_j)\Big). \tag{5.38}$$

Let us denote $b_i = \sin^2 x_i$, then we obtain g as a symmetric polynomial in b_i:

$$g = \prod_{1\le i<j\le n}\Big(y^2 - 2(b_i + b_j - 2b_ib_j)y + (b_i - b_j)^2\Big), \tag{5.39}$$

and $\{a_i\}$ are the basis of the symmetric polynomials in b_i:

$$a_i = \sum_{1\le j_1<\cdots<j_i\le n} b_{j_1}\cdots b_{j_i}. \tag{5.40}$$

Thus g can be expressed in terms of the coefficients $\{a_i\}$ of f with integer coefficients. For example:

$$n = 2\ :\quad g = y^2 - 2(a_1 - 2a_2)y + a_1^2 - 4a_2, \tag{5.41}$$

$$\begin{aligned}
&n = 3\ :\\
&\quad g = y^6 - 4(a_1 - a_2)y^5 + 2(3a_1^2 - a_2 - 4a_1a_2 - 12a_3 + 8a_1a_3)y^4\\
&\qquad - 2(2a_1^3 - a_1a_2 - 13a_3 - 2a_1^2a_2 - 4a_2^2 - 2a_1a_3 + 32a_2a_3 - 32a_3^2)y^3\\
&\qquad + (a_1^4 + 2a_1^2a_2 - 7a_2^2 - 24a_1a_3 - 8a_1a_2^2 - 16a_1^2a_3 + 120a_2a_3 + 16a_1a_2a_3 - 144a_3^2)y^2\\
&\qquad - 2(a_1^3a_2 - 3a_1a_2^2 - 9a_1^2a_3 + 27a_2a_3 - 2a_2^3 - 2a_1^3a_3 + 18a_1a_2a_3 - 54a_3^2)y\\
&\qquad + a_1^2a_2^2 - 4a_2^3 - 4a_1^3a_3 + 18a_1a_2a_3 - 27a_3^2.
\end{aligned} \tag{5.42}$$

If f is of rational coefficients, so is g.

Here are some explicit forms of $P^{\Delta_M}_{BC_r,\,s}(k_1,k_2|x)$ for lower rank r (see also [9]):

$$P^{\Delta_M}_{BC_2,\,s}(k_1,k_2|x) = \frac{4(1+k_2)(1+k_1+k_2)}{(1+k_1+2k_2)(2+k_1+2k_2)^2} - \frac{4(1+k_2)(1+k_1+k_2)}{(1+k_1+2k_2)(2+k_1+2k_2)}y + y^2, \tag{5.43}$$

$$
\begin{aligned}
P^{\Delta_M}_{BC_3,\,s}(k_1,k_2|x) \;=\;& \frac{108(1+k_2)(2+k_2)^2(1+k_1+k_2)(2+k_1+k_2)^2}{(2+k_1+2k_2)^2(3+k_1+2k_2)^3(4+k_1+2k_2)^4}\\
&-\frac{108(1+k_2)(2+k_2)^2(1+k_1+k_2)(2+k_1+k_2)^2(10+3k_1+6k_2)}{(2+k_1+2k_2)^2(3+k_1+2k_2)^3(4+k_1+2k_2)^4}y\\
&+\frac{9(2+k_2)^2(2+k_1+k_2)^2}{(2+k_1+2k_2)^2(3+k_1+2k_2)^2(4+k_1+2k_2)^4}\big(164+196k_1+41k_1^2\\
&\qquad +392k_2+292k_1k_2+32k_1^2k_2+292k_2^2+96k_1k_2^2+64k_2^3\big)\,y^2\\
&-\frac{4(2+k_2)(2+k_1+k_2)}{(2+k_1+2k_2)^2(3+k_1+2k_2)^2(4+k_1+2k_2)^3}\big(792+1278k_1+639k_1^2\\
&\qquad +99k_1^3+2556k_2+3088k_1k_2+1052k_1^2k_2+88k_1^3k_2+3088k_2^2\\
&\qquad +2562k_1k_2^2+504k_1^2k_2^2+16k_1^3k_2^2+1708k_2^3+832k_1k_2^3+64k_1^2k_2^3\\
&\qquad +416k_2^4+80k_1k_2^4+32k_2^5\big)\,y^3\\
&+\frac{6(2+k_2)(2+k_1+k_2)(26+17k_1+34k_2+8k_1k_2+8k_2^2)}{(2+k_1+2k_2)(3+k_1+2k_2)(4+k_1+2k_2)^2}y^4\\
&-\frac{12(2+k_2)(2+k_1+k_2)}{(3+k_1+2k_2)(4+k_1+2k_2)}y^5+y^6.
\end{aligned}
\tag{5.44}
$$

5.2.6 $\mathcal{R}=\Delta$ for D_r

These are the $k_1=0$ and $k_2=0$ limit of the formulae given in the previous subsection.

$$P^{\Delta_+}_{D_4,\,c2}(x) \;=\; (1+x)^3(-3/5+x)(-1/5+x^2)^4, \tag{5.45}$$

$$P^{\Delta}_{D_4,\,s2}(x) \;=\; x^6(-4/5+x^2)^8(-16/25+x^2), \tag{5.46}$$

$$P^{\Delta_+}_{D_5,\,c2}(x) \;=\; x^4(1+x)^3(-1/7+x)(-4/7+x^2)^2(-3/7+x^2)^4, \tag{5.47}$$

$$P^{\Delta}_{D_5,\,s2}(x) \;=\; (-1+x^2)^4x^6(-4/7+x^2)^8(-3/7+x^2)^4(-48/49+x^2), \tag{5.48}$$

$$P^{\Delta_+}_{D_6,\,c2}(x) \;=\; 3^{-9}\,7^{-7}(1+x)^4(-3-14x+21x^2)(1-14x^2+21x^4)^4(7-54x^2+63x^4)^2, \tag{5.49}$$

$$P^{\Delta}_{D_6,\,s2}(x) \;=\; 3^{-18}\,7^{-14}x^8(8-28x^2+21x^4)^8(16-72x^2+63x^4)^4(128-560x^2+441x^4). \tag{5.50}$$

It is trivial to verify that (3.10) are satisfied:

$$P^{\Delta}_{D_r,\,s2}(x) = P^{\Delta_+}_{D_r,\,c2}(u)P^{\Delta_+}_{D_r,\,c2}(-u)\Big|_{u^2\to 1-x^2}. \tag{5.51}$$

The Dynkin diagram folding $D_{r+1}\to B_r$ relates the functions

$$P^{\Delta_{M+}}_{BC_r,\,c2}(2,0|x)\left(P^{\Delta_{S+}}_{BC_r,\,c2}(2,0|x)\right)^2 = P^{\Delta_+}_{D_{r+1},\,c2}(x) = P^{\Delta_{M+}}_{BC_{r+1},\,c2}(0,0|x), \tag{5.52}$$

which is the trigonometric counterpart of the identity (4.37).

Next we discuss the systems based on the exceptional root systems. As in the Calogero systems, we have relied on numerical evaluation of the equilibrium points.

5.3 E_r

5.3.1 $\mathcal{R} = \mathbf{27}$ and Δ for E_6

We have evaluated two polynomials independently:

$$
\begin{aligned}
P^{\mathbf{27}}_{E_6,\,c2}(x) &= \prod_{\mu\in\mathbf{27}} \Big(x - \cos(2\mu\cdot\bar{q})\Big) \\
&= \frac{(-1+x)^3(1+2x)^6}{2^{18}\,7^4\,11^6} \big(-743 - 42651\,x + 708939\,x^2 - 1704045\,x^3 - 1890504\,x^4 \\
&\qquad +7043652\,x^5 + 1260336\,x^6 - 9391536\,x^7 + 4174016\,x^9\big)^2, \qquad (5.53)
\end{aligned}
$$

and

$$
\begin{aligned}
P^{\mathbf{27}}_{E_6,\,s2}(x) &= \prod_{\mu\in\mathbf{27}} \Big(x - \sin(2\mu\cdot\bar{q})\Big) \\
&= \frac{x^3(-3+4x^2)^3}{2^{18}\,7^4\,11^6} \big(-221709312 + 39409774992\,x^2 - 786312492840\,x^4 \\
&\qquad +6804048466593\,x^6 - 32072860850184\,x^8 + 89147361696624\,x^{10} \\
&\qquad -149154571577088\,x^{12} + 147001580732160\,x^{14} - 78400843057152\,x^{16} \\
&\qquad +17422409568256\,x^{18}\big). \qquad (5.54)
\end{aligned}
$$

Although the set of minimal weights $\mathbf{27}$ is not *even*, that is $-\mathbf{27} = \overline{\mathbf{27}} \neq \mathbf{27}$, these two polynomials are related. The "formula (3.10)" is valid,

$$
P^{\mathbf{27}}_{E_6,\,s2}(x) = \sqrt{P^{\mathbf{27}}_{E_6,\,c2}(u)}\sqrt{P^{\mathbf{27}}_{E_6,\,c2}(-u)}\bigg|_{u^2\to 1-x^2}. \qquad (5.55)
$$

This is the same situation encountered in the D_5 (conjugate) spinor representations $\mathbf{S}$ ($\bar{\mathbf{S}}$) in (5.33). This provides a strong support for the above results.

As for $\mathcal{R} = \Delta$, we have:

$$
\begin{aligned}
P^{\Delta_+}_{E_6,\,c2}(x) &= \prod_{\rho\in\Delta_+} \Big(x - \cos(2\rho\cdot\bar{q})\Big) \\
&= \frac{(2x+1)^6}{2^{24}\,7^7\,11^{11}} \big(-235 - 627\,x + 231\,x^2 + 847\,x^3\big) \\
&\quad \times \big(-743 - 42651\,x + 708939\,x^2 - 1704045\,x^3 - 1890504\,x^4 \\
&\qquad +7043652\,x^5 + 1260336\,x^6 - 9391536\,x^7 + 4174016\,x^9\big)^3, \qquad (5.56)
\end{aligned}
$$

$$
P^{\Delta}_{E_6,\,s2}(x) = \prod_{\rho\in\Delta_+} \Big(x^2 - \sin(2\rho\cdot\bar{q})\Big)
$$

$$
\begin{aligned}
&= \frac{(-3+4\,x^2)^6}{2^{48}\,7^{14}\,11^{22}}\,\left(-48384+422928\,x^2-1036728\,x^4+717409\,x^6\right) \qquad (5.57)\\
&\quad \times \left(-221709312+39409774992\,x^2-786312492840\,x^4+6804048466593\,x^6\right.\\
&\qquad -32072860850184\,x^8+89147361696624\,x^{10}-149154571577088\,x^{12}\\
&\qquad \left.+147001580732160\,x^{14}-78400843057152\,x^{16}+17422409568256\,x^{18}\right)^3 .
\end{aligned}
$$

5.3.2 $\mathcal{R} = \mathbf{56}$ for E_7

We have evaluated two polynomials independently:

$$
\begin{aligned}
P^{\mathbf{56}_+}_{E_7,\,c2}(x) &= \prod_{\mu\in\mathbf{56}_+}\Big(x-\cos(2\mu\cdot\bar{q})\Big)\\
&= \frac{x^4}{11^4\,13^5\,17^6}\,\left(9332954265600-345319307827200\,x^2+5422446428313600\,x^4\right.\\
&\qquad -47902580312348160\,x^6+266584469614182720\,x^8\\
&\qquad -991356255189780480\,x^{10}+2543382104409514368\,x^{12}\\
&\qquad -4564307435286703104\,x^{14}+5717674981551733200\,x^{16}\\
&\qquad -4899020276961851040\,x^{18}+2736363552042360240\,x^{20}\\
&\qquad \left.-897719270582318184\,x^{22}+131214258464743597\,x^{24}\right), \qquad (5.58)
\end{aligned}
$$

and

$$
\begin{aligned}
P^{\mathbf{56}}_{E_7,\,s2}(x) &= \prod_{\mu\in\mathbf{56}}\Big(x-\sin(2\mu\cdot\bar{q})\Big)\\
&= \frac{(-1+x^2)^4}{11^8\,13^{10}\,17^{12}}\,\left(7824285157-1019921980260\,x^2+44927774191218\,x^4\right.\\
&\qquad -933762748148260\,x^6+10512912980210355\,x^8\\
&\qquad -70729109671077000\,x^{10}+302444017343367900\,x^{12}\\
&\qquad -850322103495681960\,x^{14}+1590230624766864795\,x^{16}\\
&\qquad -1957192223677842580\,x^{18}+1521592634309937618\,x^{20}\\
&\qquad \left.-676851830994604980\,x^{22}+131214258464743597\,x^{24}\right)^2 . \qquad (5.59)
\end{aligned}
$$

These two polynomials satisfy (3.10),

$$
P^{\mathbf{56}}_{E_7,\,s2}(x) = P^{\mathbf{56}_+}_{E_7,\,c2}(u)\,P^{\mathbf{56}_+}_{E_7,\,c2}(-u)\Big|_{u^2\to 1-x^2} . \qquad (5.60)
$$

5.3.3 $\mathcal{R}=\Delta$ for E_7 and E_8

The polynomials $P^{\Delta}_{E_r,s(2)}(x)$, $P^{\Delta_+}_{E_r,c2}(x)$, $r=7,8$ are too long to be displayed here. Their degrees are 63 and 126 for E_7 and 120 and 240 for E_8. They are given in [9]. They all satisfy the consistency condition (3.10)

$$P^{\Delta}_{E_r,\,s2}(x)=\left.P^{\Delta_+}_{E_r,\,c2}(u)P^{\Delta_+}_{E_r,\,c2}(-u)\right|_{u^2\to 1-x^2}\qquad (r=6,7,8). \tag{5.61}$$

at the level of each factor.

5.4 F_4

We present the polynomials as a function of $k\equiv g_S/g_L$. The polynomials $P^{\Delta_{L+},\Delta_{S+}}_{F_4,\,c2}(k|x)$ and $P^{\Delta_L,\Delta_S}_{F_4,\,s2}(k|x)$, satisfying the condition (3.10), are too lengthy to be displayed here. They are given in [9]. Here we give $P^{\Delta_L,\Delta_S}_{F_4,\,s}(k|x)$ which have shorter forms. As before we use $y=x^2$.

5.4.1 $\mathcal{R}=\Delta_L$ for F_4

$$\begin{aligned}
&P^{L}_{4,\,s}(k|y)\equiv P^{\Delta_L}_{F_4,\,s}(k|x)=\prod_{\rho\in\Delta_L}\Big(x-\sin(\rho\cdot\bar{q})\Big)=\prod_{\rho\in\Delta_{L+}}\Big(y-\sin^2(\rho\cdot\bar{q})\Big)\\
&=\frac{2^{12}3^6(1+k)^6(2+k)^2(3+k)^3(1+2k)}{(3+2k)^3(4+3k)^4(5+3k)^5(6+5k)^6}\\
&\quad-\frac{2^{13}3^6(1+k)^6(2+k)^2(3+k)^3(1+2k)(14+9k)}{(3+2k)^3(4+3k)^4(5+3k)^5(6+5k)^6}\,y\\
&\quad+\frac{2^{11}3^6(1+k)^5(2+k)^2(3+k)^3(1+2k)(232+346k+123k^2)}{(3+2k)^2(4+3k)^4(5+3k)^5(6+5k)^6}\,y^2\\
&\quad-\frac{2^{11}3^4(1+k)^5(2+k)(3+k)^3}{(3+2k)^2(4+3k)^4(5+3k)^5(6+5k)^6}\\
&\qquad\times\left(30432+133672k+211560k^2+155726k^3+54075k^4+7128k^5\right)\,y^3\\
&\quad+\frac{2^{8}3^6(1+k)^4(2+k)(3+k)^2}{(3+2k)^2(4+3k)^3(5+3k)^4(6+5k)^6}\\
&\qquad\times\left(19296+90360k+159652k^2+137582k^3+61155k^4+13264k^5+1088k^6\right)\,y^4\\
&\quad-\frac{2^{9}3^4(1+k)^4(2+k)(3+k)^2}{(3+2k)^2(4+3k)^3(5+3k)^4(6+5k)^6}\big(283824+1395972k+2711556k^2\\
&\qquad\qquad+2704381k^3+1489217k^4+447066k^5+65952k^6+3456k^7\big)\,y^5\\
&\quad+\frac{2^{7}3^4(1+k)^3(3+k)^2}{(3+2k)^2(4+3k)^3(5+3k)^3(6+5k)^6}\big(1046592+6283632k+15907184k^2\\
&\qquad\qquad+22205264k^3+18708264k^4+9754573k^5+3088726k^6
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad +553392k^7+47232k^8+1152k^9\big)\,y^6 \\
&-\frac{2^8 3^4(1+k)^3(3+k)^2}{(3+2k)^2(4+3k)^2(5+3k)^3(6+5k)^5}\big(35736+163412k+300546k^2 \\
&\qquad\qquad\qquad +286499k^3+151260k^4+43412k^5+6048k^6+288k^7\big)\,y^7 \\
&+\frac{864(1+k)^2(3+k)}{(3+2k)(4+3k)^2(5+3k)^2(6+5k)^4} \\
&\quad\times\big(33120+130392k+199564k^2+150034k^3+57649k^4+10632k^5+720k^6\big)\,y^8 \\
&-\frac{1152(1+k)^2(3+k)}{(3+2k)(4+3k)^2(5+3k)^2(6+5k)^3} \\
&\quad\times\big(3312+10668k+12946k^2+7313k^3+1899k^4+180k^5\big)\,y^9 \\
&+\frac{144(1+k)^2(3+k)(116+133k+30k^2)}{(3+2k)(4+3k)(5+3k)(6+5k)^2}y^{10} \\
&-\frac{72(1+k)(3+k)}{(5+3k)(6+5k)}y^{11}+y^{12}.
\end{aligned}
\tag{5.62}
$$

5.4.2 $\mathcal{R}=\Delta_S$ for F_4

$$
P^S_{4,s}(k|y)\equiv P^{\Delta_S}_{F_4,s}(k|x)=\prod_{\rho\in\Delta_S}\Big(x-\sin(\rho\cdot\bar{q})\Big)=\prod_{\rho\in\Delta_{S+}}\Big(y-\sin^2(\rho\cdot\bar{q})\Big)
$$

$$
\begin{aligned}
=&\ \frac{729k^3(1+k)^6(2+k)(3+k)(1+2k)^2}{(3+2k)^2(4+3k)^3(5+3k)^3(6+5k)^5} \\
&-\frac{2916k^2(1+k)^6(2+k)(3+k)(1+2k)^2(9+7k)}{(3+2k)^2(4+3k)^3(5+3k)^3(6+5k)^5}y \\
&+\frac{1458k(1+k)^5(2+k)(3+k)(1+2k)^2(48+115k+58k^2)}{(3+2k)^2(4+3k)^2(5+3k)^3(6+5k)^5}\,y^2 \\
&-\frac{324(1+k)^5(3+k)(1+2k)}{(3+2k)^2(4+3k)^2(5+3k)^3(6+5k)^5} \\
&\quad\times\big(1152+11712k+33125k^2+38811k^3+20104k^4+3804k^5\big)\ y^3 \\
&+\frac{729(1+k)^4(1+2k)(1536+8960k+17519k^2+15049k^3+5788k^4+804k^5)}{(3+2k)(4+3k)^2(5+3k)^3(6+5k)^4}\ y^4 \\
&-\frac{324(1+k)^4(1+2k)}{(3+2k)(4+3k)^2(5+3k)^3(6+5k)^4} \\
&\quad\times\big(26496+112704k+177478k^2+130823k^3+45354k^4+5913k^5\big)\ y^5 \\
&+\frac{162(1+k)^3}{(3+2k)(4+3k)^2(5+3k)^3(6+5k)^3} \\
&\quad\times\big(37824+208304k+455436k^2+505691k^3+300828k^4+90935k^5+10902k^6\big)\ y^6 \\
&-\frac{324(1+k)^3(9984+42832k+70360k^2+55311k^3+20783k^4+2978k^5)}{(3+2k)(4+3k)^2(5+3k)^2(6+5k)^3}\,y^7
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{54(1+k)^2(4224+13765k+16027k^2+7876k^3+1380k^4)}{(3+2k)(4+3k)(5+3k)^2(6+5k)^2}\,y^8 \\
&-\frac{72(1+k)^2(2+k)(345+628k+276k^2)}{(3+2k)(4+3k)(5+3k)(6+5k)^2}\,y^9 \\
&+\frac{36(1+k)^2(52+29k)}{(4+3k)(5+3k)(6+5k)}\,y^{10}-\frac{36(1+k)}{(6+5k)}\,y^{11}+y^{12}
\end{aligned}
\tag{5.63}
$$

The folding $E_6 \to F_4$ relates E_6 polynomials to F_4 polynomials at the coupling ratio $k \equiv g_S/g_L = 2$. We have corresponding to (4.47)

$$
P^{\Delta_S}_{F_4,\,s2}(2|x) = P^{\mathbf{27}}_{E_6,\,s2}(x)/x^3, \quad P^{\Delta_S}_{F_4,\,c2}(2|x) = P^{\mathbf{27}}_{E_6,\,c2}(x)/(x-1)^3, \tag{5.64}
$$

$$
P^{\Delta_L}_{F_4,\,a}(2|x)\left(P^{\Delta_S}_{F_4,\,a}(2|x)\right)^2 = P^{\Delta}_{E_6,\,a}(x) \quad (a = s, s2), \tag{5.65}
$$

$$
P^{\Delta_{L+}}_{F_4,\,c2}(2|x)\left(P^{\Delta_{S+}}_{F_4,\,c2}(2|x)\right)^2 = P^{\Delta_+}_{E_6,\,c2}(x). \tag{5.66}
$$

The self-duality of the F_4 Dynkin diagram relates Δ_L polynomials to Δ_S ones. For example, we obtain:

$$
\frac{847\,P^{L}_{4,\,s}(2|y)}{847y^3-1386y^2+594y-27} = \frac{64\,P^{S}_{4,\,s}(2|y)}{(4y-3)^3}, \tag{5.67}
$$

$$
\frac{717409\,P^{L}_{4,\,s2}(2|y)}{717409y^3-1036728y^2+422928y-48384} = \frac{64\,P^{S}_{4,\,s2}(2|y)}{(4y-3)^3}, \tag{5.68}
$$

$$
\frac{847\,P^{\Delta_{L+}}_{F_4,\,c2}(2|x)}{847x^3+231x^2-627x-235} = \frac{8\,P^{\Delta_{S+}}_{F_4,\,c2}(2|x)}{(2x+1)^3}, \tag{5.69}
$$

which are a factor of the parent polynomials $P^{\Delta}_{E_6,\,s}$, $P^{\Delta}_{E_6,\,s2}$ and $P^{\Delta_+}_{E_6,\,c2}$, respectively.

5.5 G_2

Two types of polynomials $\prod_{\rho\in\mathcal{R}_+}(x-\cos(2\rho\cdot\bar{q}))$ and $\prod_{\rho\in\mathcal{R}}(x-\sin(2\rho\cdot\bar{q}))$ are evaluated. For the latter we use $y = x^2$.

5.5.1 $\mathcal{R} = \Delta_L$ for G_2

$$
\begin{aligned}
P^{\Delta_{L+}}_{G_2,\,c2}(k|x) &= \prod_{\rho\in\Delta_{L+}}\Big(x-\cos(2\rho\cdot\bar{q})\Big) \\
&= \frac{27-81k-99k^2+107k^3+80\,k^4-16\,k^5}{2(2+k)^2\,(3+2\,k)^3} + \frac{3(27-81\,k^2-40\,k^3+16\,k^4)}{2(2+k)(3+2\,k)^3}\,x \\
&\quad +\frac{3(3+2\,k-2\,k^2)}{(2+k)(3+2\,k)}\,x^2+x^3,
\end{aligned}
\tag{5.70}
$$

$$
\begin{aligned}
P^{L}_{2,\,s2}(k|y) \;\equiv\; P^{\Delta_L}_{G_2,\,s2}(k|x) &= \prod_{\rho\in\Delta_L}\Big(x-\sin(2\rho\cdot\bar q)\Big) = \prod_{\rho\in\Delta_{L+}}\Big(y-\sin^2(2\rho\cdot\bar q)\Big)\\
&= -\frac{729\,(1+k)^2\,(-3+k+8\,k^2)^2}{4\,(2+k)^4\,(3+2\,k)^5}\\
&\quad+\frac{729\,(1+k)^2\,(6+13\,k+8\,k^2)\,(9-6\,k+13\,k^2+8\,k^3)}{4\,(2+k)^3\,(3+2\,k)^6}\,y\\
&\quad-\frac{27\,(1+k)\,(9+12\,k+13\,k^2+8\,k^3)}{(2+k)^2\,(3+2\,k)^3}\,y^2+y^3, \qquad (5.71)
\end{aligned}
$$

5.5.2 $\mathcal{R}=\Delta_S$ for G_2

$$
\begin{aligned}
P^{\Delta_{S+}}_{G_2,\,c2}(k|x) &= \prod_{\rho\in\Delta_{S+}}\Big(x-\cos(2\rho\cdot\bar q)\Big)\\
&= \frac{-9-21\,k-13\,k^2+k^3}{2(2+k)(3+2\,k)^2}+\frac{3(-3-4\,k+k^2)}{2(2+k)(3+2\,k)}\,x+\frac{3\,k}{3+2\,k}\,x^2+x^3, \qquad (5.72)
\end{aligned}
$$

$$
\begin{aligned}
P^{S}_{2,\,s2}(k|y) \;\equiv\; P^{\Delta_S}_{G_2,\,s2}(k|x) &= \prod_{\rho\in\Delta_S}\Big(x-\sin(2\rho\cdot\bar q)\Big) = \prod_{\rho\in\Delta_{S+}}\Big(y-\sin^2(2\rho\cdot\bar q)\Big)\\
&= -\frac{27\,(-3+k)^2\,k\,(1+k)^2}{4\,(2+k)\,(3+2\,k)^4}+\frac{27\,(1+k)^2\,(9+12\,k+k^2+2\,k^3)}{4\,(2+k)^2(3+2\,k)^3}\,y\\
&\quad-\frac{9\,(1+k)\,(3+2\,k+k^2)}{(2+k)\,(3+2\,k)^2}\,y^2+y^3. \qquad (5.73)
\end{aligned}
$$

They satisfy the formula (3.10)

$$
P^{\Delta_L}_{G_2,\,s2}(x) = P^{\Delta_{L+}}_{G_2,\,c2}(u)P^{\Delta_{L+}}_{G_2,\,c2}(-u)\Big|_{u^2\to1-x^2}\ ,\quad P^{\Delta_S}_{G_2,\,s2}(x) = P^{\Delta_{S+}}_{G_2,\,c2}(u)P^{\Delta_{S+}}_{G_2,\,c2}(-u)\Big|_{u^2\to1-x^2}\ . \qquad (5.74)
$$

The Dynkin diagram folding $D_4\to G_2$ implies

$$
P^{\Delta_{S+}}_{G_2,\,c2}(3|x) = P^{\mathcal{R}_+}_{D_4,\,c2}(x)/(x-1),\quad P^{\Delta_S}_{G_2,\,s2}(3|x) = P^{\mathcal{R}}_{D_4,\,s2}(x)/x^2\quad (\mathcal{R}=\mathbf{V},\mathbf{S},\bar{\mathbf{S}}), \qquad (5.75)
$$

$$
P^{\Delta_L}_{G_2,\,a}(3|x)\left(P^{\Delta_S}_{G_2,\,a}(3|x)\right)^3 = P^{\Delta}_{D_4,\,a}(x)\quad (a=s,s2), \qquad (5.76)
$$

$$
P^{\Delta_{L+}}_{G_2,\,c2}(3|x)\Big(P^{\Delta_{S+}}_{G_2,\,c2}(3|x)\Big)^3 = P^{\Delta_+}_{D_4,\,c2}(x), \qquad (5.77)
$$

which correspond to (4.51). The self-duality of the G_2 Dynkin diagram relates $P^{\Delta_L}_{G_2,\,s(2)}(3|x)$ and $P^{\Delta_S}_{G_2,\,s(2)}(3|x)$ (see [9] for $P^{\Delta_{L,S}}_{G_2,\,s}(k|x)$):

$$
\frac{5P^{L}_{2,\,s}(3|y)}{5y-1} = \frac{P^{S}_{2,\,s}(3|y)}{y-1},\qquad \frac{25P^{L}_{2,\,s2}(3|y)}{25y-16} = \frac{P^{S}_{2,\,s2}(3|y)}{y}, \qquad (5.78)
$$

$$
\frac{5P^{\Delta_{L+}}_{G_2,\,c2}(3|x)}{5x-3} = \frac{P^{\Delta_{S+}}_{G_2,\,c2}(3|x)}{x+1}, \qquad (5.79)
$$

which are a factor of the parent polynomials, $P^{\Delta}_{D_4,\,s}$, $P^{\Delta}_{D_4,\,s2}$ and $P^{\Delta_+}_{D_4,\,c2}$, respectively.

6 Summary and Comments

We have derived Coxeter (Weyl) invariant polynomials associated with equilibrium points in Calogero and Sutherland systems based on all root systems. For the classical root systems, the polynomials are well-known classical orthogonal polynomials; Hermite, Laguerre, Chebyshev and Jacobi of the degree equal to the rank r of the root system ($r+1$ for the A_r case), when the smallest set of weights $\mathcal{R}$ are chosen. For the other choices of $\mathcal{R}$'s, the polynomials are related with the corresponding classical polynomials but they no longer form an orthogonal set. For the exceptional and non-crystallographic root systems, these polynomials are new. Some polynomials are given in [9], since they are too lengthy to be displayed in this paper. These new polynomials have (much) higher degree than the rank r; 27 and 36 for E_6, 28 and 63 for E_7, 120 for E_8, 12 for F_4, 3 for G_2, m for $I_2(m)$, 15 for H_3 and 60 for H_4. Defined only for sporadic degrees, these new polynomials do not have the orthogonality property, except for those corresponding to the dihedral group $I_2(m)$ with the uniform coupling $g = g_e = g_o$. In this case Chebyshev polynomials are obtained [1].

All these new polynomials share one remarkable property with the classical polynomials; their coefficients are rational functions of the ratio of the coupling constants with all integer coefficients. In most cases, they are monic polynomials with integer coefficients only.

In the rest of this section, we give a heuristic argument for "deriving" the classical orthogonal polynomials with the proper weight function from the pre-potential W (2.4) at equilibrium. We add one degree of freedom, a new coordinate q_{r+1} (q_{r+2} for A_r), to the rank r system at equilibrium:

$$W(q_1, \ldots, q_r) \to \widetilde{W}(q_{r+1}) = W(\bar{q}_1, \ldots, \bar{q}_r, q_{r+1}), \tag{6.1}$$

and consider (rescaled) q_{r+1} as the new variable. This is allowed only for the classical root systems in which r can be any positive integer. Since $\mathbf{V}$ of A_{r+2} has one more element μ_{r+2} than that of A_r, and Δ_S of B_{r+1} (BC_{r+1}) has two more elements $\mathbf{e}_{r+1}$ and $-\mathbf{e}_{r+1}$ than that of B_r (BC_r), we multiply $\sqrt{dq_{r+2}}$ for A_r case, $(\sqrt{dq_{r+1}})^2 = dq_{r+1}$ for B_r (BC_r) case, see (6.4), (6.7), (6.10) and (6.13).

6.1 Hermite

The pre-potential for the A_r Calogero system is

$$W = -\frac{1}{2}\omega q^2 + g \sum_{1\le j<l\le r+1} \log(q_j - q_l).$$

After rescaling

$$q_{r+2} = \sqrt{\frac{g}{\omega}} z, \tag{6.2}$$

we obtain from (6.1)

$$\widetilde{W}(z)/g = -\frac{1}{2}z^2 + \sum_{j=1}^{r+1} \log(z - \sqrt{\frac{\omega}{g}}\bar{q}_j) + (z\text{-indep.}). \tag{6.3}$$

If we extract a function $\psi_{r+1}(z)$ from

$$\begin{aligned} e^{\widetilde{W}/g}\sqrt{dq_{r+2}} &= (z\text{-indep.}) \times e^{-z^2/2}\prod_{j=1}^{r+1}(z - \sqrt{\frac{\omega}{g}}\bar{q}_j) \times \sqrt{dz} \\ &= (z\text{-indep.}) \times e^{-z^2/2} H_{r+1}(z)\sqrt{dz} \\ &= \psi_{r+1}(z)\sqrt{dz}, \end{aligned} \tag{6.4}$$

it satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} dz\, \psi_n(z)\psi_m(z) \propto \delta_{n,m}.$$

6.2 Laguerre

The pre-potential for the B_r Calogero system is

$$W = -\frac{1}{2}\omega q^2 + g_L \sum_{1\le j<l\le r} \log\Big((q_j - q_l)(q_j + q_l)\Big) + g_S \sum_{j=1}^{r} \log q_j.$$

After rescaling

$$q_{r+1} = \sqrt{\frac{g_L}{\omega}} z, \tag{6.5}$$

we obtain from (6.1)

$$\widetilde{W}(z)/g_L = -\frac{1}{2}z + \sum_{j=1}^{r} \log\Big(z - \Big(\sqrt{\frac{\omega}{g_L}}\bar{q}_j\Big)^2\Big) + \frac{k}{2}\log z + (z\text{-indep.}), \quad k \equiv g_S/g_L. \tag{6.6}$$

If we extract a function $\psi_r(z)$ from

$$
\begin{aligned}
e^{\widetilde{W}/g_L} dq_{r+1} &= (z\text{-indep.}) \times z^{k/2} e^{-z/2} \prod_{j=1}^{r} \Big(z - \Big(\sqrt{\frac{\omega}{g_L}} \bar{q}_j \Big)^2 \Big) \times z^{-1/2} dz \\
&= (z\text{-indep.}) \times z^{\alpha/2} e^{-z/2} L_r^{(\alpha)}(z) dz \\
&= \psi_r(z) dz, \qquad \alpha \equiv k-1, \qquad (6.7)
\end{aligned}
$$

it satisfies the orthogonality relation

$$
\int_0^\infty dz\, \psi_n(z) \psi_m(z) \propto \delta_{n,m}.
$$

6.3 Chebyshev

This is slightly contrived. The pre-potential for the A_r Sutherland system is

$$
W = g \sum_{1 \le j < l \le r+1} \log \sin(q_j - q_l).
$$

By the choice of $\bar{q}$ (5.1) and its property $\bar{q}_j = -\bar{q}_{r+2-j}$, after defining

$$
\sin q_{r+2} = z, \tag{6.8}
$$

we obtain from (6.1)

$$
\widetilde{W}(z)/g = \sum_{j=1}^{r+1} \log(z - \sin \bar{q}_j) + (z\text{-indep.}). \tag{6.9}
$$

If we extract a function $\psi_{r+1}(z)$ from

$$
\begin{aligned}
e^{\widetilde{W}/g} \sqrt{dq_{r+2}} &= (z\text{-indep.}) \times \prod_{j=1}^{r+1} (z - \sin \bar{q}_j) \times (1 - z^2)^{-1/4} \sqrt{dz} \\
&= (z\text{-indep.}) \times (1 - z^2)^{-1/4} T_{r+1}(z) \sqrt{dz} \\
&= \psi_{r+1}(z) \sqrt{dz}, \qquad (6.10)
\end{aligned}
$$

it satisfies the orthogonality relation

$$
\int_{-1}^{1} dz\, \psi_n(z) \psi_m(z) \propto \delta_{n,m}.
$$

6.4 Jacobi

The pre-potential for the BC_r Sutherland system is

$$W = g_M \sum_{1\le j<l\le r} \log\Big(\sin(q_j - q_l)\sin(q_j + q_l)\Big) + \sum_{j=1}^{r}(g_S \log\sin q_j + g_L \log\sin 2q_j).$$

After defining z by

$$\cos 2q_{r+1} = z, \tag{6.11}$$

we obtain from (6.1) ($k_1 \equiv g_S/g_M$, $k_2 \equiv g_L/g_M$)

$$\widetilde{W}(z)/g_M = \sum_{j=1}^{r} \log(z - \cos 2\bar{q}_j) + \frac{k_1+k_2}{2}\log(1-z) + \frac{k_2}{2}\log(1+z) + (z\text{-indep.}). \tag{6.12}$$

If we extract a function $\psi_r(z)$ from

$$\begin{aligned} e^{\widetilde{W}/g_M}dq_{r+1} &= (z\text{-indep.}) \times (1-z)^{(k_1+k_2)/2}(1+z)^{k_2/2}\prod_{j=1}^{r}(z-\cos 2\bar{q}_j) \times (1-z^2)^{-1/2}dz \\ &= (z\text{-indep.}) \times (1-z)^{\alpha/2}(1+z)^{\beta/2}P_r^{(\alpha,\beta)}(z)dz \\ &= \psi_r(z)dz, \qquad\qquad \alpha \equiv k_1+k_2-1, \quad \beta \equiv k_2 - 1, \end{aligned} \tag{6.13}$$

it satisfies the orthogonality relation

$$\int_{-1}^{1} dz\, \psi_n(z)\psi_m(z) \propto \delta_{n,m}.$$

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